§7

7.1 $\Rightarrow$ Suppose $N \trianglelefteq G$. Then (by Def^n 7.1) $N^g = N$ $\forall g \in G$. $\therefore$ $g^{-1}ng \in N$ $\forall n \in N$ and $\forall g \in G$.

$\Leftarrow$ Now suppose $g^{-1}ng \in N$ $\forall n \in N$ and $\forall g \in G$.

$\therefore$ $N^x \subseteq N$ $\forall x \in G$ $(*)$.

Let $g \in G$ — we show that $N^g = N$. By $(*)$ $N^g \subseteq N$. Also by $(*)$ $N^{g^{-1}} \subseteq N$. Hence

$$N = N^1 = N^{(g^{-1}g)} = (N^{g^{-1}})^g \subseteq N^g \text{ (using q 3.1).}$$

Thus $N^g = N$ and so $N \trianglelefteq G$ (by Def^n 7.1).

7.2 (i) Lemma 4.7 $\Rightarrow$ $Z(G) \neq 1$. Since $G \neq Z(G)$ (G is not abelian), $|Z(G)| = p$ or p^2 by Lagrange's theorem. If $|Z(G)| = p^2$, then $|G/Z(G)| = p$, and so $G/Z(G) \cong \mathbb{Z}_p$. In particular, $G/Z(G)$ is cyclic. By Lemma 7.6 G is abelian, a contradiction. So $|Z(G)| \neq p^2$. $\therefore$ $|Z(G)| = p$.

(ii) Let $g \in G \setminus Z(G)$. Note that $Z(G) \subseteq C_G(g)$.

Since $C_G(g) \leqslant G$, $|C_G(g)| = p, p^2$ or p^3 by Lagranges' theorem. If $|C_G(g)| = p$, then $Z(G) = C_G(g)$ which is impossible as $g \in C_G(g)$ and $g \notin Z(G)$. If $|C_G(g)| = p^3$, then $C_G(g) = G$ and so $g \in Z(G)$ whereas $g \notin Z(G)$.

$\therefore$ $|C_G(g)| = p^2$ and so $|g^G| = [G : C_G(g)]$

$$= \frac{|G|}{|C_G(g)|} = \frac{p^3}{p^2} = p.$$

Now $Z(G)$ consists of p conjugacy classes (as $|Z(G)| = p$) and, using (ii), $G \setminus Z(G)$ consists of $\frac{p^3 - p}{p} = p^2 - 1$ conjugacy classes.

$\therefore$ G has $p + p^2 - 1$ conjugacy classes.

7.3 $$HN = \bigcup_{h \in H} hN \underset{\substack{\uparrow \\ \text{Lemma 7.2(iv) as } N \trianglelefteq G}}{=} \bigcup_{h \in H} Nh = NH.$$

$\therefore$ $NH \leqslant G$ by Lemma 3.5.

7.4 (i) Since $N_1 \leqslant G$ and $N_2 \leqslant G$, $N_1 \cap N_2 \leqslant G$. Let $g \in G$ and $n \in N_1 \cap N_2$. Since $N_i \trianglelefteq G$, $g^{-1}ng \in N_i$ $(i=1,2)$ by Lemma 7.2 (iii) (see q 7.1)

$\therefore$ $g^{-1}ng \in N_1 \cap N_2$ and hence $N_1 \cap N_2 \trianglelefteq G$ by Lemma 7.2 (iii).

(ii) By q 7.3 $N_1N_2 \leqslant G$. Let $g \in G$ and $n \in N_1N_2$. Then $n = n_1n_2$ for some $n_1 \in N_1$ and $n_2 \in N_2$. Now

$$g^{-1}ng = g^{-1}n_1n_2g = g^{-1}n_1gg^{-1}n_2g \in N_1N_2$$

since $g^{-1}n_1g \in N_1$, $g^{-1}n_2g \in N_2$ (as $N_1 \trianglelefteq G$, $N_2 \trianglelefteq G$).

$\therefore$ $N_1N_2 \trianglelefteq G$ by Lemma 7.2 (iii).

7.5 (i) Let $h \in H$ and $n \in H \cap N$. Since $N \trianglelefteq G$, $h^{-1}nh \in N$. Because $H \leqslant G$ and $n, h \in H$, $h^{-1}nh \in H$ also. $\therefore$ $h^{-1}nh \in H \cap N$. We already have $H \cap N \leqslant H$ and so $H \cap N \trianglelefteq H$ by Lemma 7.2 (iii).

(ii) Already know $C_G(N) \leqslant G$. Let $g \in G$ and $c \in C_G(N)$. Aim to show $g^{-1}cg \in C_G(N)$.

Let n be an arbitrary element of N.

Since $N^g = N$, $n = g^{-1}n_1 g$ for some $n_1 \in N$.

Consider

$$\begin{aligned}(g^{-1}cg)n &= g^{-1}cgg^{-1}n_1 g \\ &= g^{-1}cn_1 g \\ &= g^{-1}n_1 c g \quad (\text{as } c \in C_G(N) \text{ and } n_1 \in N) \\ &= g^{-1}n_1 g g^{-1}cg \\ &= n(g^{-1}cg)\end{aligned}$$

$\therefore$ $g^{-1}cg \in C_G(N)$ and so $N \trianglelefteq G$ by Lemma 7.2(iii).

7.6 (i)

	$\overline{(1)}$	$\overline{(123)}$	$\overline{(234)}$	$\overline{(1234)}$	$\overline{(12)}$	$\overline{(23)}$
$\overline{(1)}$	$\overline{(1)}$	$\overline{(123)}$	$\overline{(234)}$	$\overline{(1234)}$	$\overline{(12)}$	$\overline{(23)}$
$\overline{(123)}$	$\overline{(123)}$	$\overline{(234)}$	$\overline{(1)}$	$\overline{(12)}$	$\overline{(23)}$	$\overline{(1234)}$
$\overline{(234)}$	$\overline{(234)}$	$\overline{(1)}$	$\overline{(123)}$	$\overline{(23)}$	$\overline{(1234)}$	$\overline{(12)}$
$\overline{(1234)}$	$\overline{(1234)}$	$\overline{(23)}$	$\overline{(12)}$	$\overline{(1)}$	$\overline{(234)}$	$\overline{(123)}$
$\overline{(12)}$	$\overline{(12)}$	$\overline{(1234)}$	$\overline{(23)}$	$\overline{(123)}$	$\overline{(1)}$	$\overline{(234)}$
$\overline{(23)}$	$\overline{(23)}$	$\overline{(12)}$	$\overline{(1234)}$	$\overline{(234)}$	$\overline{(123)}$	$\overline{(1)}$

(ii) By (i) G/N is not abelian, and so, as $|G/N| = 6$, $G/N \cong S_3$ (by HINT).

(iii) Three subgroups of G/N of order 2 are:- $\langle \overline{(1234)} \rangle$, $\langle \overline{(12)} \rangle$ and $\langle \overline{(23)} \rangle$. The three subgroups of G of order 8 containing N are:-

$$\overline{(1)} \cup \overline{(1234)} = \{(1), (12)(34), (13)(24), (14)(23), (1234), (13), (1432), (24)\} = \langle (1234), (13) \rangle.$$

$$\overline{(1)} \cup \overline{(12)} = \{(1), (12)(34), (13)(24), (14)(23), (12), (34), (1324), (1423)\} = \langle (1324), (34) \rangle$$

$$\overline{(1)} \cup \overline{(23)} = \{(1), (12)(34), (13)(24), (14)(23), (23), (1342), (1243), (14)\} = \langle (1243), (23) \rangle$$

7.7 (i) $x^2 = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in N \therefore$ order $\bar{x} = 2$

$y^2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in N \therefore$ order $\bar{y} = 2$

$z^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in N \therefore$ order $\bar{z} = 2$

$w^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \notin N$

$w^3 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in N \; \therefore$ order $\bar{w} = 3$

(ii) Partial multiplication :-

	$\bar{1}$	$\bar{x}$	$\bar{y}$	$\bar{z}$
$\bar{1}$	$\bar{1}$	$\bar{x}$	$\bar{y}$	z
$\bar{x}$	$\bar{x}$	$\bar{1}$	$\bar{z}$	$\bar{y}$
$\bar{y}$	$\bar{y}$	$\bar{z}$	1	$\bar{x}$
$\bar{z}$	$\bar{z}$	$\bar{y}$	$\bar{x}$	$\bar{1}$

$\Rightarrow \{\bar{1}, \bar{x}, \bar{y}, \bar{z}\}$ is a subgroup of G/N.

(Example of calculations :-

$xy = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$ and $\overline{\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}} = \overline{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$

as $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. So $\bar{x}\bar{y} = \bar{z}$.

(iii) $\bar{w}^{-1}\bar{x}\bar{w} = \overline{w^{-1}xw}$. Now

$w^{-1}xw = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = y$

$\therefore \bar{w}^{-1}\bar{x}\bar{w} = \bar{y}$.

$\bar{w}^{-2}\bar{x}\bar{w}^2 = \overline{w^{-2}xw^2}$ (note $w^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ so $w^{-2} = w$)

$w^{-2}xw^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

Now $\overline{\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}} = \overline{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$ as $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$\therefore \bar{w}^{-2}\bar{x}\bar{w}^2 = \bar{z}$

(iv) $G/N \cong A_4$.

7.8 (Notation as in question and HINTS)

$\phi : G/N \longrightarrow G/M$ defined by $\bar{g}\phi = \tilde{g}$.

(a) If $\bar{g} = \bar{g}_1$, then $gg_1^{-1} \in N$ (by Theorem 1.5(ii)).
So $gg_1^{-1} \in M$ since $N \leq M$. Hence $\tilde{g} = \tilde{g}_1$ (by Theorem 1.5(ii)).

$\therefore \bar{g}\phi = \tilde{g} = \tilde{g}_1 = \bar{g}_1\phi$.

(b) $\ker\phi = \{\bar{g} \in G/N \mid \bar{g}\phi = 1_{G/M}\} =$
$\{\bar{g} \in G/N \mid \tilde{g} = \tilde{1}\} = \{\bar{g} \in G/N \mid g \in M\} = M/N$.

(c) Let $\tilde{g} \in G/M$ (so $g \in G$). Then $\bar{g} \in G/N$ and $\bar{g}\phi = \tilde{g}$. Thus the image of ϕ is G/M.

So $(G/N)/(M/N) = (G/N)/\ker\phi \underset{\text{FIRST ISOMORPHISM THEOREM}}{\cong} \operatorname{im}\phi = G/M$

$\therefore (G/N)/(M/N) \cong G/M$.