§6

6.1 (i) is true (by Lemma 6.1)

(ii) is false (by Lemma 6.1)

(iii) is true (in general $A \times B \cong B \times A$)

(iv) is false as the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ either have order 1 or 2, whereas $\mathbb{Z}_4 \times \mathbb{Z}_4$ has elements of order 4.

6.2 (i) $\mathbb{Z}_{10} \times \mathbb{Z}_{15} \times \mathbb{Z}_{20} \cong$

$\mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_5$ (using Lemma 6.1)

$\cong \mathbb{Z}_5 \times (\mathbb{Z}_2 \times \mathbb{Z}_5) \times (\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5)$

$\cong \mathbb{Z}_5 \times \mathbb{Z}_{10} \times \mathbb{Z}_{60}$ (using Lemma 6.1)

$\therefore$ torsion coefficients are 5, 10, 60.

(ii) $\mathbb{Z}_{28} \times \mathbb{Z}_{42} \cong \mathbb{Z}_4 \times \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7$

$\cong (\mathbb{Z}_2 \times \mathbb{Z}_7) \times (\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_7) \cong \mathbb{Z}_{14} \times \mathbb{Z}_{84}$

(using Lemma 6.1)

$\therefore$ torsion coefficients are 14, 84.

(iii) $\mathbb{Z}_9 \times \mathbb{Z}_{14} \times \mathbb{Z}_6 \times \mathbb{Z}_{16} \cong$

$\mathbb{Z}_9 \times \mathbb{Z}_2 \times \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_{16}$

$\cong \mathbb{Z}_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_3) \times (\mathbb{Z}_7 \times \mathbb{Z}_9 \times \mathbb{Z}_{16})$

$\cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1008}$ (using Lemma 6.1)

$\therefore$ torsion coefficients are 2, 6, 1008.

6.3 (i) Since $|G| = 9$, by Theorem 6.2 either $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $G \cong \mathbb{Z}_9$. Calculating we see that the orders of the elements of G are either 1 or 3. $\therefore G \ncong \mathbb{Z}_9$, and so $G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. (Note $\mathbb{Z}_9$ only has two elements of order 3, so we only really need check the orders of three non-identity elements of G.)

(ii) Since $|G| = 24$, by Theorem 6.2 either $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_{24}$. Calculating shows that 8 (an element of G) has order 12. $\therefore G \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ ($\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$

has no elements of order 12). Also 109 and 134 both have order 2. Since $\mathbb{Z}_{24}$ has only one element of order 2, $G \not\cong \mathbb{Z}_{24}$. $\therefore$ $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$.

6.4 $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$; $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$; $\mathbb{Z}_2 \times \mathbb{Z}_{36}$; $\mathbb{Z}_3 \times \mathbb{Z}_{24}$; $\mathbb{Z}_6 \times \mathbb{Z}_{12}$; $\mathbb{Z}_{72}$.

6.5 (i) Since 1 has order 1, $1 \in T$ and so $\emptyset \neq T \subseteq G$.

Let $x, y \in T$. Then $x^n = 1$ and $y^m = 1$ for some $n, m \in \mathbb{N} \cup \{0\}$. Hence $(y^{-1})^m = 1$.

Now $(xy^{-1})^{mn} \underset{\uparrow}{=} x^{mn}(y^{-1})^{mn} = (x^n)^m((y^{-1})^m)^n$

$\boxed{\text{BECAUSE } G \text{ IS ABELIAN}}$

$= 1^m 1^n = 1.$

$\therefore$ xy^{-1} has finite order and so $xy^{-1} \in T$.

Hence T is a subgroup of G by the subgroup criterion.

(ii) NOT ALWAYS. COUNTEREXAMPLE:—

Take $G = \mathbb{Z}_2 \times \mathbb{Z}$ $(\mathbb{Z}_2 = \{0, 1\})$

Let $n \in \mathbb{Z}$ with $n \neq 0$. Then $(1, n)$ and $(0, -n)$ are elements of G, both of infinite order. So $(1,n), (0,-n) \in B = \{x \in G \mid x \text{ has infinite order or } x = 1\}$.

But $(1,n)(0,-n) = (1,0)$ which has order 2 and so $(1,n)(0,-n) \notin B$. $\therefore B \not\leq G$.

6.6 Using Theorem 6.2

$k=1$ $\quad \mathbb{Z}_p$

$k=2$ $\quad \mathbb{Z}_{p^2}, \mathbb{Z}_p \times \mathbb{Z}_p$.

$k=3$ $\quad \mathbb{Z}_{p^3}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

$k=4$ $\quad \mathbb{Z}_{p^4}, \mathbb{Z}_{p^3} \times \mathbb{Z}_p, \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$,

$\mathbb{Z}_{p^2} \times \mathbb{Z}_p \times \mathbb{Z}_p, \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$.

So there are 5 pairwise non-isomorphic abelian groups of order p^4.