§5

5.1 Note that $Xg = \{g_1g, g_2g, \ldots, g_kg\}$ is a $k$-element subset of $G$. So $Xg \in \Omega$.

Let $X \overset{=\{g_1,\ldots,g_k\}}{\in} \Omega$ and $g, h \in G$. Then

$$(Xg)h = \{g_1g, g_2g, \ldots, g_kg\}h$$
$$= \{(g_1g)h, (g_2g)h, \ldots, (g_kg)h\}$$

(using the definition twice)

and $X(gh) = \{g_1(gh), g_2(gh), \ldots, g_k(gh)\}$

(by definition)

Since group multiplication is associative,

$$(Xg)h = X(gh).$$

Also $X1 = \{g_1 1, g_2 1, \ldots, g_k 1\}$
$$= \{g_1, g_2, \ldots, g_k\} = X.$$

$\therefore \Omega$ is a $G$-set.

5.2 (i) $\Omega$ is a $G$-orbit (so $G$ is transitive on $\Omega$).

(ii) $G$ has 3 orbits on $\Omega$:-

$\{1,2,3,4,5,6\}$, $\{7,8,9,10,11,12,13,14\}$, $\{15\}$.

5.3 (i) Since $\sigma = (1,2,3,\ldots,n) \in S_n$, applying $\sigma$ we see that

$$\Omega \subseteq \{1g \mid g \in S_n\} \subseteq \Omega.$$

So $\Omega$ is an $S_n$-orbit. $\therefore$ $S_n$ is transitive on $\Omega$.

(ii) Let $\alpha \in \Omega$ with $\alpha \neq 1$. Since $n \geq 3$, we may choose $\beta \in \Omega$ with $\alpha \neq \beta \neq 1$.

Then $\sigma = (1,\alpha,\beta) \in A_n$ ($(1,\alpha,\beta)$ is an even permutation) and $1\sigma = \alpha$.

$$\therefore \Omega \subseteq \{1g \mid g \in A_n\} \subseteq \Omega$$

So $\Omega$ is an $A_n$-orbit. $\therefore$ $A_n$ is transitive on $\Omega$.

5.4 Since $\Omega$ is a $G$-orbit,

$$|G| = |\Omega||G_x| \quad \text{by Lemma}$$

By hypothesis $\Omega \setminus \{x\}$ is a $G_x$-orbit and so, using Lemma again

$$|G_x| = |\Omega \setminus \{x\}||(G_x)_y| \quad \text{where } y \text{ is}$$

some element of $\Omega \setminus \{x\}$.

$$= (|\Omega| - 1)|(G_x)_y|.$$

$$\therefore |G| = |\Omega|(|\Omega| - 1)|(G_x)_y|$$

$\Rightarrow |\Omega|(|\Omega| - 1)$ divides $|G|$.

5.5 Burnsides theorem (Theorem 5.9) gives (here $t = 1$)

$$|G| = \sum_{g \in G} |fix_{\Omega}(g)|$$

$$= |\Omega| + \sum_{\substack{g \in G \\ g \neq 1}} |fix_{\Omega}(g)| \quad (*)$$

(note $\Omega = fix_{\Omega}(1)$)

Since $|\Omega| > 1$, if $|\mathrm{fix}_\Omega(g)| \geq 1$ $\forall g \in G$, then we get a contradiction to $(*)$. $\therefore \exists$ $g \in G$ s.t. $|\mathrm{fix}_\Omega(g)| = 0$. So there exist elements of $G$ having no fixed points on $\Omega$.

5.6 Let $G$ act upon $\Omega = G$ via conjugation.

For $g \in G$,

$$\begin{aligned} \mathrm{fix}_\Omega(g) &= \{x \in \Omega \mid \overset{\text{ACTION}}{x^g} = x\} \\ &= \{x \in G \mid g^{-1}xg = x\} \\ &= C_G(g) \end{aligned}$$

An orbit of $G$ on $\Omega = G$ is just a conjugacy class of $G$. $\therefore$ the number of $G$-orbits on $\Omega = G$ is $k$.

Substituting into Theorem (Burnside's theorem) gives

$$k = \frac{1}{|G|} \sum_{g \in G} |C_G(g)|$$

$$\Rightarrow k|G| = \sum_{g \in G} |C_G(g)|.$$