§4

4.1 This will follow from: for $m \in \mathbb{N}$,

$x^m = 1 \iff y^m = 1.$

Since x and y are conjugate in G, $\exists\, g \in G$ s.t. $g^{-1}xg = y$.

Suppose $x^m = 1$. Then $y^m = (g^{-1}xg)^m =$

$\underbrace{(g^{-1}xg)(g^{-1}xg)\ldots(g^{-1}xg)}_{m \text{ times}} = g^{-1}\underbrace{xx\ldots x}_{m \text{ times}}g$

$= g^{-1}x^m g = g^{-1}1g = 1.$

Suppose $y^m = 1$. Then $(g^{-1}xg)^m = y^m = 1$, So

$\underbrace{(g^{-1}xg)(g^{-1}xg)\ldots(g^{-1}xg)}_{m \text{ times}} = 1 \Rightarrow g^{-1}x^m g = 1$

$\Rightarrow x^m = g1g^{-1} = 1.$

4.2 (i) $\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}$

($\mathbb{Z}_6$ is abelian so all its conjugacy classes have just one element.)

(ii) $\{I\}, \{-I\}, \{J, -J\}, \{K, -K\}$,

$\{L, -L\}$ (five conjugacy classes

(iii) $\{(1)\}$, $\{(12)(34), (13)(24), (14)(23)\}$, $\{(123), (214), (341), (432)\}$, $\{(132), (241), (314), (423)\}$ (four conjugacy classes)

4.3 By the class equation

$$|G| = n_1 + n_2 + \ldots + n_k$$

where k is even (n_i the sizes of the conjugacy classes). If all the n_i are odd, then their sum will be even, and so $|G|$ is even.

If, say n_j is even for some j, then, as $n_j \mid |G|$ we also get that $|G|$ is even.

4.4 Suppose $Z(G) = 1$ and argue for a contradiction. Using the class equation gives

$$|G| = 1 + n_2 + n_3 + \ldots + n_k$$

where $n_i > 1$ for all i, $2 \leq i \leq k$.

Since $n_i > 1$, $\exists$ a prime q_i s.t. $q_i \mid n_i$. Because $n_i \mid |G|$, we get $q_i \mid |G|$. $\therefore$, as p

is the smallest prime divisor of $|G|$,

$$p \leq q_i \leq n_i \text{ for all } i,\ 2 \leq i \leq k.$$

$$\therefore |G| = 1 + n_2 + n_3 + \ldots + n_k \geq 1 + (k-1)p$$

$$\Longrightarrow \frac{|G|}{p} \geq \frac{1}{p} + (k-1)$$

Since $\frac{|G|}{p} \in \mathbb{N}$, this gives $\frac{|G|}{p} \geq k$, contrary to the hypothesis $k > \frac{|G|}{p}$. Thus we deduce

$Z(G) \neq 1$.

4.5 $|G| = \sum_{i=1}^{k} n_i = 1 + n_2 + \ldots + n_k$ (CLASS EQUATION)

(i) $k = 2 \Rightarrow |G| = 1 + n_2$ $(*)$. Since $n_2 \mid |G|$, $(*)$ implies $n_2 \mid 1$, and so $n_2 = 1$. $\therefore |G| = 2$ and so $G \cong \mathbb{Z}_2$ (by (a)).

(ii) $k = 3 \Rightarrow |G| = 1 + n_2 + n_3$ $(*)$ WLOG $n_2 \leq n_3$. Because $n_3 \mid |G|$, $(*) \Rightarrow n_3 \mid 1 + n_2$. In particular $n_3 \leq 1 + n_2$. $\therefore$ (as $n_2 \leq n_3$) either $n_2 = n_3$ or $1 + n_2 = n_3$.

Case 1: $n_2 = n_3$. $(*) \Rightarrow n_2 \mid 1$ and so $n_2 = n_3 = 1$.

$\therefore |G| = 3$ and hence $G \cong \mathbb{Z}_3$ (by (a))

Case 2: $1 + n_2 = n_3$. $(*)$ becomes $|G| = 1 + n_2 + 1 + n_2 = 2 + 2n_2$. Since $n_2 \mid |G|$ we get $n_2 \mid 2$ and so $n_2 = 1$ or 2. If $n_2 = 1$, then $|G| = 4$ and hence G is abelian which implies $k = 4$, but $k = 3$. $\therefore n_2$ cannot happen. So $n_2 = 2$ and hence $|G| = 6$. By (b) $G \cong \mathbb{Z}_6$ or S_3. Since $k = 6$ for $\mathbb{Z}_6$, we get $G \cong S_3$ (which has $k = 3$).

4.6 $\{(1)\}, \{(12)(34), (13)(24), (14)(23)\}, \{(12), (13), (14), (23), (24), (34)\}, \{(1234), (1243), (1324), (1342), (1423), (1432)\}, \{(123), (132), (124), (142), (134), (143), (234), (243)\}$ (using Lemma 4.)

4.7 $g^G = \{(12)(34)(56), (12)(35)(46), (12)(36)(45), (13)(24)(56), (13)(25)(46), (13)(26)(45), (14)(23)(56), (14)(25)(36), (14)(26)(34), (15)(23)(46), (15)(24)(36), (15)(26)(34), (16)(23)(45), (16)(24)(35), (16)(25)(34)\}$ $|g^G| = 15$

$$\frac{|G|}{|C_G(g)|} = |g^G| \Rightarrow \frac{6!}{|C_G(g)|} = 15 \Rightarrow |C_G(g)| = 48.$$