§3

3.1 By definition

$$S^{(gh)} = \{(gh)^{-1}x(gh) \mid x \in S\}$$
$$= \{h^{-1}g^{-1}xgh \mid x \in S\}, \text{ and}$$

$$(S^g)^h = \{g^{-1}xg \mid x \in S\}^h$$
$$= \{h^{-1}(g^{-1}xg)h \mid x \in S\}.$$

$$\therefore\ S^{(gh)} = (S^g)^h.$$

3.2 (i) Now $S \cup S^- = \{(123), (234), (132), (243)\}$.

Calculating gives $(123)(132) = (1)$, $(123)(234) = (13)(24)$, $(234)(123) = (12)(34)$, $(123)(243)(123) = (14)(23)$.

$\therefore\ H = \{(1), (12)(34), (13)(24), (14)(23)\} \subseteq \langle S \rangle$.

(Note $H \leq G$ because, by question 2.8, H is a subgroup of S_4. Since $G = A_4$ is a subgroup of S_4 and $H \subseteq G$, H is a subgroup of G — see §1 of lecture notes.)

Again as $\langle S \rangle$ is a subgroup of G (by Lemma 3.2), H is a subgroup of $\langle S \rangle$.

$\therefore$ $4 = |H|$ divides the order of $\langle S \rangle$ by Lagrange's theorem.

Similarly $\langle (123) \rangle = \{(1), (123), (132)\} \subseteq \langle S \rangle$ and then, as above, $\langle (123) \rangle$ is a subgroup of $\langle S \rangle$. $\therefore$ 3 divides the order of $\langle S \rangle$ by Lagrange's theorem. Hence $12 \mid |\langle S \rangle|$.

Since $|G| = \frac{4!}{2} = 12$ (question 2.9), we must have $\langle S \rangle = G$.

(ii) Since $S \leq G$ (see part (i)), we know $S \leq N_G(S) \leq G$ (§3 lecture notes).

$$\therefore\ 3 = [G:S] = [G:N_G(S)][N_G(S):S]$$

(Theorem 1.6(ii))

So $[G:N_G(S)] = 1$ or 3.

Hence either $N_G(S) = G$ or $N_G(S) = S$.

Calculating

$$(123)^{-1}(12)(34)(123) = (14)(23),$$

$$(123)^{-1}(14)(23)(123) = (13)(24),$$

$$(123)^{-1}(13)(24)(123) = (12)(34)$$

we deduce $S^{(123)} = S$ and so $(123) \in N_G(S)$.

Since $(123) \notin S$, $N_G(S) \neq S$ and $\therefore$

$N_G(S) = G$.

For $C_G(S)$ note that S is abelian, and hence $S \leq C_G(S) \leq G$. Now argue as for $N_G(S)$ above to show $S = C_G(S)$.

3.3 Let $H \leq G$ with $S \subseteq H$. Then $S^- \subseteq H$ (as $H \leq G$), and so products of elements in $S \cup S^-$ must again be in H (as $H \leq G$).

$\therefore \langle S \rangle \subseteq H$.

Hence $\langle S \rangle \subseteq \bigcap\limits_{\substack{H \leq G \\ S \subseteq H}} H$.

Since $S \subseteq \langle S \rangle$ and, by Lemma 3.2, $\langle S \rangle$ is a subgroup of G,

$$\bigcap_{\substack{H \leq G \\ S \subseteq H}} H \subseteq \langle S \rangle.$$

$$\therefore \quad \langle S \rangle = \bigcap_{\substack{H \leq G \\ S \subseteq H}} H.$$

3.4 Since $1 \in H$ (as $H \leq G$), $1 = g^{-1} 1 g \in H^g$, and so $H^g \neq \phi$. Two typical elements of H^g are $g^{-1}xg$, $g^{-1}yg$ where $x, y \in H$. Now

$$\begin{aligned}(g^{-1}xg)(g^{-1}yg)^{-1} &= g^{-1}xgg^{-1}y^{-1}g \\ &= g^{-1}xy^{-1}g,\end{aligned}$$

which belongs to H^g (as $xy^{-1} \in H$ because $H \leq G$).

$\therefore$ H^g is a subgroup of G by the subgroup criterion.

3.5 (i) Since $\det I_n = 1$, $I_n \in SL(n,F)$, so $SL(n,F) \neq \emptyset$. Note also $SL(n,F) \subseteq GL(n,F)$.

Let $A, B \in SL(n,F)$. Then $\det A = 1 = \det B$.

So $\det B^{-1} = \dfrac{1}{\det B} = 1$.

$\therefore \det(AB^{-1}) = (\det A)(\det B^{-1}) = 1$ and so

$AB^{-1} \in SL(n,F)$.

$\therefore SL(n,F) \leq GL(n,F)$ by the subgroup criterion.

(ii) Since $I_n \in O(n,F)$, $O(n,F) \neq \emptyset$.

Note $A \in O(n,F) \Rightarrow AA^T = I_n \Rightarrow (\det A)(\det A^T) = \det(AA^T)$

$\det I_n = 1$. So $\det A \neq 0$ and so A is invertible. $\therefore A^{-1} = A^T$. So $O(n,F) \subseteq GL(n,F)$.

Let $A, B \in O(n,F)$. Then $A^{-1} = A^T$ and $B^{-1} = B^T$.

Now $AB^{-1}(AB^{-1})^T = AB^T(AB^T)^T = AB^TBA^T$

$= AA^T = I_n \Rightarrow AB^{-1} \in O(n,F)$.

$\therefore O(n,F) \leq GL(n,F)$ by the subgroup criterion.