§1

1.1 Write each of the following permutations in S_8 as a product of disjoint cycles.

(i) $\alpha: 1 \mapsto 7$, $2 \mapsto 5$, $3 \mapsto 1$, $4 \mapsto 4$, $5 \mapsto 6$, $6 \mapsto 2$, $7 \mapsto 8$, $8 \mapsto 3$

(ii) $\beta: 1 \mapsto 2$, $2 \mapsto 4$, $3 \mapsto 6$, $4 \mapsto 3$, $5 \mapsto 7$, $6 \mapsto 1$, $7 \mapsto 8$, $8 \mapsto 5$

(iii) α^{-1} (iv) β^{-1}

(v) $\alpha\beta$ (vi) $\beta\alpha$

(vii) $\alpha^{-1}\beta^{-1}\alpha\beta$.

1.2 In S_5, mark the following statements as true or false (with reasons).

(i) $(12345)=(34512)$ (ii) $(1234)=(1342)$

(iii) $(123)=(123)(4)(5)$ (iv) $(1)=(5)$

(v) $(123)(45) \neq (45)(312)$.

1.3 For $H \leq G$ write down all the right cosets of H (as subsets of G) in the following cases.

(i) $G = S_3$, $H = \{(1), (13)\}$

(ii) $G = S_3$, $H = \{(1), (123), (132)\}$

(iii) $G = S_4$, $H = \{(1), (234), (243), (23), (24), (34)\}$

Suppose G is a group and $g \in G$. Recall that g has order n means that n is the smallest natural number such that $g^n = 1$.

1.4 In S_7, what are the orders of

(i) $(12457)(36)$ (ii) $(173)(265)$ (iii) (7654321)?

1.5 If $\sigma \in S_n$ is a cycle of length r, what is the order of σ?

1.6 Let $\sigma \in S_n$ and $\sigma = \sigma_1 \sigma_2 \ldots \sigma_t$ where $\sigma_1, \sigma_2, \ldots, \sigma_t$ are (pairwise) disjoint cycles. Prove that the order of σ is the least common multiple of the lengths of $\sigma_1, \sigma_2, \ldots, \sigma_t$.

1.7 What are the orders of the permutations in question 1.1?

§2

2.1 (i) For each of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times S_3$ list the elements and determine the order of each of the elements.

(ii) Is $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$?

(iii) Is $\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$?

2.2 Let $G = \{1, 2, 4, 5, 7, 8\}$ with the binary operation being multiplication modulo 9. (Example (2.3) with $n = 9$.) Work out the order of each of the elements of G. Is G cyclic?

2.3 Let $n \in \mathbb{N}$ with $n > 1$, and set $G = \{x \in \{1, 2, \ldots, n-1\} \mid \mathrm{hcf}(x, n) = 1\}$. With $*$ being multiplication modulo n, prove that $(G, *)$ is a group. (HINT: for establishing (G4) recall

that if $x \in G$, then $\exists\ \lambda, \mu \in \mathbb{Z}$ such that
$$\lambda x + \mu n = hcf(x, n) = 1.)$$

2.4 Let $G = \{ A \in GL(2,3) \mid \det A = 1 \}$ (you may assume G is a subgroup of $GL(2,3)$).
Verify that the following are elements of G and determine their orders:- (i) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

(ii) $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$; (iii) $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$; (iv) $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$.

Prove that $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ is the only element of G of order 2.

2.5 Prove that, for $n \in \mathbb{N}$ and $q = p^m$ (where p is a prime and $m \in \mathbb{N}$),
$$|GL(n,q)| = (q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{n-1}).$$
(HINT: a square matrix is invertible $\Longleftrightarrow$ its rows are linearly independent.)

2.6 Let $G = Dih(8)$ (Example (2.4)(ii) with $n=4$) and $H = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$. Prove that H is a subgroup of G and determine the right cosets of H in G (as subsets of G).

2.7 Write the following permutations of S_8 as a product of transpositions and determine whether the permutations are odd or even.

(i) $(1376)(2548)$ (ii) $(12473)(58)(6)$

(iii) $(18)(256374)$ (iv) $(1)(274)(3)(586)$.

2.8 Let $G = S_4$ and $H = \{ (1), (12)(34), (13)(24), (14)(23) \}$. Show that H is a subgroup of G.

2.9 Let $n \in \mathbb{N}$ with $n \geq 2$. Prove that $[S_n : A_n] = 2$. Hence deduce that $|A_n| = n!/2$.

2.10 List all the elements of A_4.

§3

3.1 Let S be a subset of a group G, and let $g, h \in G$. Prove that $S^{(gh)} = (S^g)^h$.

3.2 Let $G = A_4$.

(i) For $S = \{(123), (234)\}$ determine $\langle S \rangle$.

(ii) Let $S = \{(1), (12)(34), (13)(24), (14)(23)\}$.

Work out $N_G(S)$ and $C_G(S)$.

3.3 Let S be a subset of a group G. Prove that

$$\langle S \rangle = \bigcap_{\substack{H \leq G \\ S \subseteq H}} H.$$

(HINT: use the definition of $\langle S \rangle$ and the fact that, by Lemma 3.2, $\langle S \rangle$ is a subgroup of G.)

3.4 Suppose G is a group, $H \leq G$ and $g \in G$. Prove that $H^g \leq G$.
$(H^g = \{g^{-1}hg \mid h \in H\})$

3.5 Let F be a field and let $n \in \mathbb{N}$.

(i) Write $SL(n,F)$ for the set of all $n \times n$ matrices over F which have determinant 1. Prove that $SL(n,F)$ is a subgroup of $GL(n,F)$.

(ii) Write $O(n,F)$ for the set of all $n \times n$ orthogonal matrices over F. (Recall that an $n \times n$ matrix A is orthogonal $\Leftrightarrow$ $AA^T = I_n$ where A^T is the transpose of A and I_n is the $n \times n$ identity matrix.)
Prove that $O(n,F)$ is a subgroup of $GL(n,F)$.

§4

4.1 Let G be a group with $x, y \in G$. If x and y are conjugate in G, prove that x and y have the same order.

4.2 Determine the conjugacy classes (as sets) for each of the following groups:-

(i) $\mathbb{Z}_6$

(ii) Q_8 (quaternion group of order 8)

[$Q_8 = \{I, -I, J, -J, K, -K, L, -L\}$ where
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$; $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; $L = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$
and the binary operation is matrix multiplication.]

(iii) A_4

4.3 Prove that if a finite group G has an even number of conjugacy classes, then $|G|$ is even. (HINT: use the CLASS EQUATION)

4.4 Let G be a non-trivial finite group with k conjugacy classes and let p be the least prime divisor of $|G|$. If $k > \frac{|G|}{p}$, prove that $Z(G) \neq 1$
(HINT: use the CLASS EQUATION)

4.5 Suppose G is a finite group. Prove that
(i) if G has two conjugacy classes, then $G \cong \mathbb{Z}_2$.
(ii) if G has three conjugacy classes, then $G \cong \mathbb{Z}_3$ or S_3. (ASSORTED HINTS: use the CLASS EQUATION to find the possibilities for $|G|$; use the results (a) if $|G| = p$, p a prime, then $G \cong \mathbb{Z}_p$ and (b) if $|G| = 6$, then G is isomorphic to either $\mathbb{Z}_6$ or S_3 (can you prove (b)?))

4.6 Determine the conjugacy classes of S_4 (as sets).

4.7 Let $G = S_6$ and $g = (12)(34)(56)$. List the elements of g^G. Hence deduce $|C_G(g)|$.

§5

5.1 Let G be a finite group and let Ω be the set of all k-element subsets of G, $k \in \mathbb{N}$ (k fixed). For $X \in \Omega$ (so $X = \{g_1, \dots, g_k\}$, $g_i \in G$) and $g \in G$ define

$$Xg = \{g_1 g, g_2 g, \dots, g_k g\}.$$

Verify that Ω is a G-set.

5.2 Let $\Omega = \{1, 2, \dots, 15\}$ and $G = \langle g_1, g_2, g_3 \rangle \leqslant S_\Omega$. Determine the G-orbits of Ω where the permutations g_i are as follows.

(i) $g_1 = (1,2)(3,4)(5,6)(7,8)$

$g_2 = (1,5,7,9)(10,11,12)(13,14,15)$

$g_3 = (1,2)(3,11)(4,13,15,7)(10,8,12)$.

(ii) $g_1 = (1,2,3)(4,5,6)(7,8,9)(10,11,12)$

$g_2 = (1,4)(2,6)(3,5)(9,8,11,13,14)$

$g_3 = (1,2,3,4,5)(7,9,10,12,11)(13,14)$.

5.3 Prove that

(i) S_n is transitive on $\Omega = \{1, 2, \ldots, n\}$.

(ii) if $n \geq 3$, then A_n is transitive on $\Omega = \{1, 2, \ldots, n\}$.

($G \leq S_n$. G is transitive on $\Omega = \{1, 2, \ldots, n\}$ just means Ω is a G-orbit.)

5.4 Let Ω be a G-set where G is a finite group and Ω a finite set. Suppose that G is transitive on Ω and G_x is transitive on $\Omega \setminus \{x\}$ where x is some element of Ω. Prove that $|\Omega|(|\Omega|-1)$ divides $|G|$.

5.5 Let Ω be a transitive G-set where G is a finite group and Ω a finite set.

Show that if $|\Omega| > 1$, then there exist elements of G that have no fixed points on Ω. (That is there exists $g \in G$ such that $\mathrm{fix}_{\Omega}(g) = \phi$.)

5.6 Show that if G is a finite group, then

$$\sum_{g \in G} |C_G(g)| = k|G|,$$

where k is the number of conjugacy classes of G.

§ 6

6.1 Mark the following as true or false (with reasons).

(i) $\mathbb{Z}_3 \times \mathbb{Z}_7 \cong \mathbb{Z}_{21}$ (ii) $\mathbb{Z}_4 \times \mathbb{Z}_{10} \cong \mathbb{Z}_{40}$

(iii) $\mathbb{Z}_6 \times \mathbb{Z}_{12} \cong \mathbb{Z}_{12} \times \mathbb{Z}_6$ (iv) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4 \times \mathbb{Z}_4$.

6.2 Find the torsion coefficients for each of the following abelian groups.

(i) $\mathbb{Z}_{10} \times \mathbb{Z}_{15} \times \mathbb{Z}_{20}$ (ii) $\mathbb{Z}_{28} \times \mathbb{Z}_{42}$

(iii) $\mathbb{Z}_9 \times \mathbb{Z}_{14} \times \mathbb{Z}_6 \times \mathbb{Z}_{16}$.

6.3 For the following (abelian) groups G determine the isomorphism type (as in Theorem 6.2).

(i) $G = \{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ with binary operation being multiplication mod 91.

(ii) $G = \{1, 8, 17, 19, 26, 28, 37, 44, 46, 53, 62, 64, 71, 73, 82, 89, 91, 98, 107, 109, 116, 118, 127, 134\}$ with binary operation being multiplication mod 135.

6.4 List, up to isomorphism, all abelian groups of order 72.

6.5 Suppose G is an abelian group. Let $T = \{x \in G \mid x \text{ has finite order}\}$ and $B = \{x \in G \mid x \text{ has infinite order or } x = 1\}$.

(i) Prove that T is a subgroup of G.

(ii) Is B a subgroup of G?

6.6 Let p be a prime. Show there are k (pairwise) non-isomorphic abelian groups of order p^k for $k = 1, 2, 3$. How many (pairwise) non-isomorphic abelian groups are there of order p^4?

§7

7.1 Suppose that $N \leqslant G$. Prove that

$N \trianglelefteq G \iff g^{-1}ng \in N \quad \forall n \in N$ and $\forall g \in G$.

7.2 Let p be a prime. Suppose G is a non-abelian p-group with $|G| = p^3$. Prove that

(i) $|Z(G)| = p$; and

(ii) for $g \in G \setminus Z(G)$, $|g^G| = p$.

Hence deduce that G has $p^2 + p - 1$ conjugacy classes.

7.3 Suppose G is a group. If $N \trianglelefteq G$ and $H \leqslant G$, prove that $NH \leqslant G$.

7.4 Suppose G is a group, $N_1 \trianglelefteq G$ and $N_2 \trianglelefteq G$. Prove that

(i) $N_1 \cap N_2 \trianglelefteq G$; and (ii) $N_1 N_2 \trianglelefteq G$.

7.5 Suppose G is a group, $H \leqslant G$ and $N \trianglelefteq G$.

Prove that (i) $H \cap N \trianglelefteq H$; and

(ii) $C_G(N) \trianglelefteq G$.

7.6 Let $G = S_4$ and $N = \{(1), (12)(34), (13)(24), (14)(23)\}$. (Recall that $N \trianglelefteq G$.) From lectures

$$G/N = \{\overline{(1)}, \overline{(123)}, \overline{(234)}, \overline{(1234)}, \overline{(12)}, \overline{(23)}\}.$$

(i) Work out the multiplication table of G/N.

(ii) Prove that $G/N \cong S_3$. (HINT:- use the result that a group of order 6 is isomorphic to either $\mathbb{Z}_6$ or S_3.)

(iii) Find the three subgroups of G of order 8 which contain N. (HINT: Lemma 7.9)

7.7 Let $G = SL(2,3)$ and $N = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right\rangle$. (Note that $N \leqslant Z(G)$.) Set $x = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}$; $y = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$; $z = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; $w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$; and set $\bar{g} = Ng$ (as usual).

(i) In G/N calculate the orders of $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{w}$;

(ii) prove that $\{\bar{1}, \bar{x}, \bar{y}, \bar{z}\}$ is a subgroup of G/N;

(iii) what is $\bar{w}^{-1}\bar{x}\bar{w}$ and $\bar{w}^{-2}\bar{x}\bar{w}^{2}$?; and

(iv) can you identify G/N?

7.8 Suppose G is a group, $N \trianglelefteq G$, $M \trianglelefteq G$ and $N \leq M$ (from lectures $M/N \trianglelefteq G/N$). Prove that

$$(G/N)/(M/N) \cong G/M.$$

[HINTS: Notation: $\bar{g} = Ng$ and $\tilde{g} = Mg$ $(g \in G)$

$\therefore$ $G/N = \{\bar{g} \mid g \in G\}$ and $G/M = \{\tilde{g} \mid g \in G\}$.

Define $\phi : G/N \longrightarrow G/M$ by $\bar{g}\phi = \tilde{g}$.

Show that (a) ϕ is well-defined.

(b) $\ker \phi = M/N$.

(c) image of ϕ is G/M.

Hence deduce the result.]

§8

8.1 Suppose G is a finite simple group. If G is abelian, prove that $G \cong \mathbb{Z}_q$ for some prime q.

8.2 Suppose G is a finite simple group and H a subgroup of G with $[G:H] = n > 1$.

(i) Prove that G is isomorphic to a subgroup of S_n.

(ii) Prove that $|G|$ divides $n!$.

8.3 Give a composition series for each of the following (with reasons)

(i) Dih(8) (ii) S_4 (iii) S_5

In each case what are the composition factors?

§9

9.1 (i) Find all the Sylow 2-subgroups and all the Sylow 3-subgroups of S_4.

(ii) Find all the Sylow 2-subgroups of A_5.

9.2 Suppose $G = GL(2,p)$ where p is a prime. Let $P = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| \, a \in F \right\}$ ($F = \{0, 1, \ldots, p-1\}$ the finite field with p elements). Prove that

(i) P is a Sylow p-subgroup of G;

(ii) $N_G(P) = \left\{ \begin{pmatrix} b & a \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in F, \ b \neq 0 \neq c \right\}$; and

(iii) using (ii) determine the number of Sylow p-subgroups in G.

[HINTS for (i) use q 2.5 to give $|G|$; for (ii) calculate $N_G(P)$ directly.]

9.3 Show that a normal p-subgroup of a finite group G (where p is a prime) is contained in every Sylow p-subgroup of G.

9.4 Let G be a finite group, p a prime and $P \in Syl_p G$.

(i) If $N \trianglelefteq G$, show that $P \cap N \in Syl_p N$.

(ii) If $N \trianglelefteq G$, show that $PN/N \in Syl_p G/N$.

(iii) Give an example to show that if $H \leq G$, then $P \cap H$ need not necessarily be a Sylow p-subgroup of H.

[HINT for (ii): use Theorem 7.17 to show PN/N has order $|P|/|P \cap N|$ and then use part (i)]

9.5 If G is a group and $|G| = p^2 q$ where p and q are primes, prove that G cannot be simple.

9.6 Prove that there are no simple groups of the following order:

(i) 56 ; (ii) $3^2.5.7$; (iii) $2^3.3.7.23$.