Mathieu Groups and Chamber Graphs

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Abstract

In this project, we use the geometries defined by Ronan and Stroth for the Mathieu groups, M_{12}, M_{22}, M_{23} and M_{24}, to calculate their chamber graphs in Magma. We replicate by hand the calculations for M_{12} and M_{22}, using a more combinatorial approach. The work on M_{12} and M_{22} has not appeared in the literature prior to this.

Contents

1 Introduction

Much of this project relies on the work of Jacques Tits, which was described by the Abel Committee as being of extraordinary depth and influence. The committee went on to say that, together with John Thompson, Tits forms the backbone of modern group theory.

The Classification theorem states that any finite simple group is one of the cyclic groups of prime order, alternating groups of degree at least five, groups of Lie Type, 26 sporadic groups, or the Tits group. Perhaps the most interesting of these are the sporadic groups because they fail to have a unifying description.
The concept of buildings as simplicial complexes was first developed by Tits in the 1960s, these buildings give a geometric interpretation of each group of Lie type. The sporadic groups do not give rise to buildings, but in an attempt to further understand these groups the construction of buildings has been mimicked by Ronan and Stroth.
As a result of a suggestion from Luis Puig, Tits translated the language of buildings into that of chamber graphs in 1981. The rich structure of buildings result in the associated chamber graphs displaying some rather distinctive properties. One way to gain an understanding of how accurately the constructions for the sporadic groups mirror buildings, is to form the chamber graphs of their geometries and investigate how much "Building-like" behaviour they exhibit.
The construction of the chamber graphs of M_{12}, M_{22}, M_{23} and M_{24}, along with the investigation of their "Building-like" behaviour form the main body this project.

In the next section, we briefly look at Weyl groups. However, this section only covers the essentials needed for this project. In particular we examine a running example of $Sym(4)$ viewed as a Weyl group. Perhaps it is useful to follow the advice of Tits "When reading a maths book, start at the end. If you understand it you do not need the rest." I would not necessarily recommend starting at the end but those familiar with Weyl groups could certainly miss this section.

In the section three we examine some of the theory of buildings, sometimes known as Bruhat-Tits Buildings, which provide a means of viewing certain classes of groups in a combinatorial and geometric way. After looking at the theory, we trace back through with an example that demonstrates the key definitions and shows the theorems in action. In later sections we use the geometries of Ronan and Stroth to emulate the structure of buildings for other groups, leading to some interesting results.

Section four focuses on (B, N)-pairs as developed by Tits, although they were also developed independently and concurrently by C. W. Curtis. These are of interest to us because every (B, N)-pair gives rise to a building. We can also use (B, N) pairs to calculate the Weyl group of the building that, as we shall see, has much influence on the structure of the building and its chamber graph.

Section five introduces the notations of a chamber graph. Due to the common properties of the group geometries studied in this project, we are able to make some equivalences that simplify calculations. Perhaps the most salient of these is the correspondence between B-orbits and right cosets which proves advantageous in the next section, as well as in the code for MAGMA. Our motivation for the study of chamber graphs is that, when our group geometry is a building, the chamber graph encapsulates the geometric information of the building.

In section six we study two small examples of chamber graphs of buildings. These give us an opportunity to apply some of the observations on (B, N)-pairs in section four, and to verify some theorems and comments from earlier sections. In addition, the two chamber graphs of buildings exemplify some characteristic properties, and we can use them to demonstrate the controlling natural of the Weyl group. We show some of the ways in which data from the building can be translated to and from that of the chamber graph, in such a way as to lose no information.

Mathieu groups are five of the 26 sporadic simple groups, and form the main focus of section seven. They are multiply transitive groups, through which property Emile Mathieu first discovered them between 1860 and 1873. Initially, there was much contention around the existence of these groups, in particular M_{24}, since it was not immediately clear that the generators given by Mathieu did not generate alternating group $Alt(24)$. Perhaps the clearest way to view these groups is as automorphism groups of Steiner Systems. This way of viewing Mathieu groups is due to Witt, and we repeatedly use these Steiner systems throughout this project.

The most interesting sections of this project are doubtlessly those that appertain to the original work on M_{12}. The calculations of the chamber graph by hand not only have a pleasingly elegant solution but also enable us easily to perform further calculations, such as finding maximal opposite sets. Throughout these sections we define and make extensive use of the Kitten, developed by Conway and Curtis in 1984. Although it is not used further in the project, we also mention another facet of M_{12}, the MINIMOG. Discovered before the Kitten by Conway, the MINIMOG gives an indication of the applications of the Mathieu groups to coding theory in which these groups play a role.

Mirroring the work in the previous sections, we calculate M_{22} both by MAGMA and by hand. As for M_{12}, the work for M_{22} by hand had not yet been done. Although the solution is similar to that of M_{12} in its combinatorial approach, the methods used are sufficiently different to be of independent interest. Here we use a much more classical group theoretic approach. In particular, we calculate generators of B and make considerable use of the orbit-stabilizer theorem. In the same way that we used the Kitten for M_{12}, here we use The MOG for M_{22}. The MOG was developed in 1974 and, because it offers both an intellectually and aesthetically beautiful solution to a complex problem, is perhaps my first mathematical love.

For both M_{12} and M_{22} we play what we refer to as "The Jigsaw game", so named because we construct the chamber graph by first forming the pieces, the B-orbits, and then simply slotting them into place. The name was also chosen to echo the recurring presence of games and strategies that we encounter in the later sections of this project.

M_{23} has two different geometries which correspond to two different chamber graphs. Compared to those for M_{12} and M_{22}, the chamber graphs for M_{23} are much larger, both in terms of the number of chambers and the number of B-orbits. On account of this we calculate M_{23} using MAGMA only. In this section we also discuss how we form the system from the paper of Ronan and Stroth, upon which this work is built.

As M_{24} has already been studied in great depth we simply summarize some of the results that correspond to those we find for the other groups.

2 Weyl Groups

We begin by looking at Weyl groups and roots systems as the Weyl groups, in particular, play an important role in later sections. Although it is not apparent in this project, the Weyl group is a very important driver of what happens in a group with a (B, N)-pair. The theory in this section is from Carter's book, Simple Groups of Lie Type [4].
In order to define Weyl groups we first look at reflections.

Definition 2.1. *Let V be a vector space over $\mathbb{R}$ with positive symmetric bilinear form $(\cdot,\cdot)$. For $r \in V$, we define the hyperplane of r to be $H_r = \{x \in V \mid (x, r) = 0\}$. A reflection on V is a linear operator which sends some non zero $r \in V$ to its negative and fixes H_r point-wise. We define L_r to be the line orthogonal to H_r, passing through r.*

We can characterise these reflections by the formula below,

$$w_r(x) = x - 2\frac{(x, r)}{(r, r)}r.$$

Before we can define the Weyl group we introduce the notion of a root system.

Definition 2.2. *A system of roots of V, is a subset Φ of V such that:*

(i) Φ is a finite set which does not contain the zero vector,

(ii) $Span(\Phi) = V$,

(iii) for all $r, s \in \Phi$ we have that $w_r(s) \in \Phi$,

(iv) for all $r, s \in \Phi$ we require $2\frac{(x,r)}{(r,r)} \in \mathbb{Z}$,

(v) if $r, \lambda r \in \Phi$ with $\lambda \in \mathbb{R}$ then $\lambda = \pm 1$.

Now we can define a Weyl group.

Definition 2.3. *The Weyl group of Φ is given by $W(\Phi) = \langle w_r \mid r \in \Phi\rangle$.*

Example 2.4. *Consider $e_1, \ldots, e_n$, the standard basis vectors of $\mathbb{R}^n$.
Let $\Phi = \{e_i - e_j \mid i \neq j\}$. If $r = e_i - e_j$ then H_r is the plane spanned by all vectors with equal i and j component and w_r corresponds to interchanging the basis vectors e_i and e_j.
These vectors span the subspace of $\mathbb{R}^n$ where the sum of all components is zero, and so in this case $dim(V) = n - 1$.
We can represent each $r = e_i - e_j$ by the ordered pair (i, j) and represent the reflection $w_{e_i - e_j}$ by the transposition (ij). And so $w_{(i,j)} = (ij)$.
Hence $W(\Phi) = \langle w_{e_i-e_j} \mid 1 \leq i \neq j \leq n\rangle \cong \langle (ij) \mid 1 \leq i \neq j \leq n\rangle = Sym(n)$, and so we can view $Sym(n)$ as an example of a Weyl group.*

In later sections, we are particularly interested in $Sym(4)$. Accordingly we proceed through this section using this group as a running example.

By condition (ii) of the definition of root systems we know that $Span(\Phi) = V$, however, Φ is not linearly independent. For instance, we know that for any $r \in \Phi$ we also have $-r \in \Phi$. This motivates the following definition.

Definition 2.5. *Each root system that spans a space V with $dim(V) = m$, contains a subset Π, satisfying the following:*

(i) $\Pi = \{r_1, \dots, r_m\}$,

(ii) Given any $r \in \Phi$ we can express r as $\sum_{i=1}^{m} \lambda_i r_i$,
where either $\lambda_i \geq 0$ for all i, or $\lambda_i. \leq 0$ for all i,

We call this the fundamental system.

In our running example, the fundamental system is given by $\{(1, 2), (2, 3), (3, 4)\}$.

Lemma 2.6. *We can generate W just using the fundamental reflections.*

$$W = \langle w_r \mid r \in \Pi \rangle.$$

Indeed, in the case of $Sym(4)$, we can generate the whole of $Sym(4)$ using the transpositions (12), (23) and (34).

Weyl groups are a special example of Coxeter groups.

Definition 2.7. *Let $S = \{s_1, \dots, s_n\}$ be a set of generators. Let M be an $n \times n$ symmetric matrix with all diagonal entries equal to 1, and all off diagonal entries in $\mathbb{N} \backslash \{1\} \cup \{\infty\}$. The Coxeter group defined by M is*

$$W(M) = \langle S \mid (s_i s_j)^{m_{ij}} = 1, \textit{ provided } m_{ij} \neq \infty \rangle.$$

We call S the set of simple reflections, and say the rank is given by $|S| = n$. The pair (W, S) is the Coxeter system, and W the Coxeter group.

We can view Weyl groups as Coxeter groups by taking $S = \{w_r \mid r \in \Pi\}$.

Definition 2.8. *The positive cone of V, denoted V^+, is the subset of V satisfying:*

(i) for $v \in V^+$ and $\lambda \in \mathbb{R}$ with $\lambda > 0$, we have that $\lambda v \in V^+$,

(ii) for $v_1, v_2 \in V^+$ their sum, $v_1 + v_2$, also lies in V^+,

(iii) for all $v \in V$ either $v \in V^+$, $-v \in V^+$ or v is the zero vector.

We can now define positive roots systems.

Definition 2.9. *Given a root system Φ, we define a positive root system to be $\Phi^+ = \Phi \cap V^+$. And define the negative root system to be given by $\Phi^- = \Phi \backslash \Phi^+$.*

In our example of $Sym(4)$, the positive and negative root systems are as below:

$$\Phi^+ = \{(1, 2), (1, 3), (1, 4), (2, 3), (3, 4)\},$$

$$\Phi^- = \{(4, 3), (4, 3), (4, 1), (3, 2), (3, 1)\}.$$

Definition 2.10. *For $w \in W$ we define the length of w, $\ell(w)$, to be the smallest k such that $w = w_{r_1} w_{r_2} \cdots w_{r_k}$, for $r_j \in \Pi$. If w is the identity we set $\ell(w) = 0$.*

Finding the length of an element from first principles could be a very difficult problem to solve. Even once we have found an expression for w as above, how can we easily verify whether or not this is the shortest? Thankfully in the case of Weyl groups, there is a very neat solution to this problem.

Theorem 2.11. *Let $w \in W$, define the set $N(w) = \{x \in \Phi^+ \mid w(x) \in \Phi^-\}$. Call $N(W)$, the number of positive roots sent negative. We have that $\ell(w) = |N(w)|$.*

Theorem 2.12. *There is a unique element of $w_0 \in W$ such that $w_0(\Phi^+) = \Phi^-$. In addition w_0 is of order two.*

Combining the two previous theorems, we can see that this element will be of maximal length.

Example 2.13. *We claim that in the case of $Sym(4)$, the longest element is $(14)(23)$. This can be verified by showing that $(14)(23)$ maps the positive roots to the negative.*

$$\begin{array}{lll} (1,2) \mapsto (4,3) & (1,3) \mapsto (4,1) & (1,3) \mapsto (3,1) \\ (1,3) \mapsto (4,2) & (1,3) \mapsto (3,2) & (1,3) \mapsto (2,1) \end{array}$$

Hence $(14)(23)$ is indeed the element of longest length and because $|\Phi^+| = 6$ we know the length of $(14)(23)$ is six. We can express $(14)(23)$ as $(12)(23)(34)(23)(12)(23)$, but note that without the use of Theorem 2.12 we would not have known this was the shortest decomposition.

For the duration of the project, we return to using $(x_1, x_2, \ldots, x_n)$ to be cycle notation unless stated otherwise.

3 Buildings as Simplicial Complexes

The idea of Buildings was developed by Jacques Tits in [32], as a way to view algebraic structures such as exceptional groups of Lie type in a more geometric way. The terminology Buildings and Apartments, initially called skeletons by Tits, is due to the Bourbaki group who according to Tits in [21] found his original terminology "a shambles". Tits was an honorary member of the Bourbaki group, a group of mathematicians who collectively published under the name Nicolas Bourbaki. During this time he helped to popularize the work of Coxeter. At the age of eighteen Tits was studying multiply transitive groups which caused him to rediscover the Mathieu groups which we will cover in a later section.

3.1 Theory

Firstly we run through a lot of definitions and theorems from [30] without much in the way of motivation. However, in the next subsection, we see this theory at work within an example.

Definition 3.1. *Given a set T of d elements, we can form V containing all subsets of T equipped with the containment relation as a partial order. We call any set isomorphic to V a simplex of rank $d-1$.*

Definition 3.2. *A set Δ, is a complex if:*

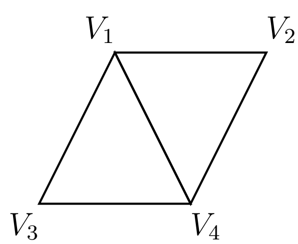
(i) *Δ is equipped with a partial order which we denote $X \subset Y$,*

(ii) *For all $X, Y \in \Delta$, there exists $Z \in \Delta$, maximal in dimension, such that $Z \subset X$ and $Z \subset Y$. We call such an element the greatest lower bound of X and Y, and denote it $X \cap Y$,*

(iii) *For all $X \in \Delta$, Δ_X given by $\{Y \in \Delta \mid Y \subset X\}$, forms a simplex.*

Definition 3.3. *A subcomplex of Δ is a subset Γ of Δ, with the property that for all $X \in \Gamma$, if $Y \in \Delta$ with $Y \subset X$ then $Y \in \Gamma$.*
We call $X \in \Delta$ an element of rank r if Δ_X is a simple of rank r. X is a vertex if it has rank one.
A complex Δ of rank d is defined to be a complex for which any element of Δ is contained in some maximal element, where the maximal elements are those of maximal rank, d.
For such a complex we say maximal elements X and Y are adjacent if $X \neq Y$ and there exists Z, a complex rank $d-1$ such that $Z \subset X$ and $Z \subset Y$.

Example 3.4. *Consider the complex drawn below, with the partial order given by inclusion.*



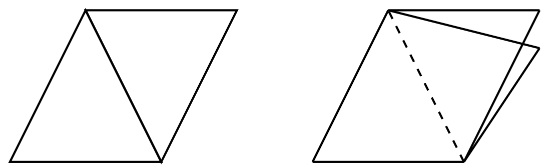
$$\Delta = \Big\{\emptyset, V_1, V_2, V_3, V_4, \{V_1, V_2\}, \{V_1, V_3\}, \{V_1, V_4\}, \{V_2, V_4\}, \{V_3, V_4\}, \{V_1, V_2, V_4\}, \{V_1, V_3, V_4\}\Big\}$$

The greatest lower bound of $\{V_1, V_3\}$ and $\{V_3, V_4\}$ is V_3.
$\Delta_{\{V_1,V_2\}} = \{\emptyset, V_1, V_2, \{V_1, V_2\}\}$, which is a simplex of rank one, and so $\{V_1, V_2\}$ is referred to as a vertex.
Our only maximal elements are $\{V_1, V_2, V_4\}$ and $\{V_1, V_3, V_4\}$, which are of rank two. These are connected by an element of rank one, namely $\{V_1, V_4\}$ therefore this is a complex of rank two.
An example of a subcomplex is $\Gamma = \{\emptyset, V_1, V_3, V_4, \{V_1, V_3\}, \{V_1, V_4\}, \{V_3, V_4\}, \{V_1, V_3, V_4\}\}$.

Definition 3.5. *A chamber complex is a complex of rank d that is connected, in other words, for all $X, Y \in \Delta$ there is a sequence of maximal elements $Z_0, \ldots Z_m$ such that $X \subset Z_0$ and $Y \in Z_m$, with Z_i and Z_{i+1} adjacent for $i = 0, .., m-1$.*

Definition 3.6. *Let Δ be a complex of rank d. We call Δ thin if every element of Δ with rank $d-1$ is contained in exactly two maximal elements. We call Δ thick if there exists some element of rank $d-1$ contained in more than 2 maximal elements.*

This can be visualised as in [22]. An element of Δ of rank $d-1$ is represented by a line, and the maximal elements by faces of the triangles. A line is contained in a maximal element if it is one of the sides of the triangle. The left diagram represents the way every two incident triangles must meet in a thin complex, in particular, every line must be shared by exactly two triangles. The right diagram represents how triangles can meet in a thick complex, the dashed line in the diagram is shared by 3 triangles.



Definition 3.7. *A building of rank d is a complex Δ of rank d, with a set of subcomplexes $\mathcal{A}$ such that:*

(i) Δ *is a thick complex,*

(ii) for all $\Sigma \in \mathcal{A}$, Σ *is a connected, thin complex of rank* d,

(iii) for all $X, Y \in \Delta$ *there exists* $\Sigma \in \mathcal{A}$ *such that* $X \in \Sigma$ *and* $Y \in \Sigma$,

(iv) Suppose $X, X' \in \Delta$, $\Sigma, \Sigma' \in \mathcal{A}$ *such that* $X, X' \in \Sigma$ *and* $X, X' \in \Sigma'$. *Then there is an isomorphism* $\phi : \Sigma \to \Sigma'$ *such that, for all* $Y \subset X$ *and for all* $Y' \subset X'$ *we have* $\phi(Y) = Y$ *and* $\phi(Y') = Y'$.

We define an apartment to be one of the subcomplexes in $\mathcal{A}$, a chamber to be a maximal element in Δ, and any element of rank $d-1$ to be a wall.

Theorem 3.8. *All apartments are isomorphic.*

Suppose we have $\{Z_i\}_{i=0}^m$, a sequence of chambers such that Z_{i-1} and Z_i have a common wall for $i = 1 : m$, and $X \subset Z_0$, $Y \subset Z_m$, for $X, Y \in \Delta$. Then we call $\{Z_i\}_{i=0}^m$ a gallery joining X and Y or length m. If there exists an i such that $Z_i = Z_{i+1}$, we say the gallery stammers. Conditions (ii) and (iii) result in any building being connected, that is for any two maximal simplices there is a gallery joining them. This enables us to define $dist(X, Y)$ to be the length of the shortest gallery between X and Y. Clearly, the shortest gallery will not stammer.

Definition 3.9. *Suppose Σ is a thin complex of rank d. A map $\sigma : \Sigma \to \Sigma$ such that $\sigma^2 = \sigma$, with the additional property that all chambers in $\sigma(\Sigma)$ have exactly 2 preimages, is called a folding.*

Lemma 3.10. *Let Δ be a building with two chambers X, Y that are adjacent in some apartment Σ. Then there exists σ_x and σ_y foldings of the apartment with the properties that:*

(i) $\sigma_x(X) = X$ *and* $\sigma_y(Y) = Y$,

(ii) $\Sigma = \sigma_x(\Sigma) \cup \sigma_y(\Sigma)$,

(iii) $\sigma_x(\Sigma) \cap \sigma_y(\Sigma)$ *contains no chambers,*

(iv) $\theta(Z) = \begin{cases} \sigma_x(Z) & \text{if } Z \in \sigma_y(\Sigma), \\ \sigma_y(Z) & \text{if } Z \in \sigma_x(\Sigma). \end{cases}$
is an order two automorphism of Σ*. We call this the reflection in the wall* $X \cap Y$*.*

Definition 3.11. *For an apartment, its Weyl group is the group generated by all the apartment's reflections.*

We give the following lemma in order to alter the definition of type given by Suzuki in [30]. In the following theorem and proof we use the term vertex in the graphical sense as opposed to simplices of rank one.

Lemma 3.12. *A finite connected graph in which every vertex has degree two contains a Hamiltonian path, that is a path that passes through each vertex exactly once.*

Proof. Let $x_0, \ldots, x_t$ be the longest path in our graph.
Suppose that there is some vertex x_s not lying on this path.
Our graph is connected and so there exists some path $x_t, y_1, \ldots, y_{a-1}, x_s$ joining x_t to x_s, of length a.
Suppose $y_i = x_j$ for some $1 \leq i \leq a-1$ and $0 \leq j \leq t-1$. Each vertex has degree 2, and so y_{i-1} is x_{j+1} or x_{j-1} with y_{i+1} being the other. Proceeding though the ys we find that this new path must be a truncation of the first one.
Both paths contain x_t as an end point, hence we find that $x_{t-1} = y_1$, $x_{t-2} = y_2$ and so on.
We know $a \leq t$ as we picked our first path to be the longest. Therefore x_s lies on the path $x_0, \ldots, x_t$. In particular, $x_s = x_{t-a}$.
This is a contradiction as we assumed x_s did not lie on the path.
Accordingly all vertices in the two paths apart from x_t are distinct.
Hence $x_0, \ldots, x_t, y_1, \ldots, y_{a-1}, x_s$ is another path.
This is another contradiction as we started by picking the longest path.
Consequently we can assume there is a path that passes through every vertex exactly once.
□

We now define the type of objects of a building as follows.

Definition 3.13. *Suppose we have a buildings* Δ*, containing an apartment* Σ*. Consider a chamber* X *of* Σ*, and a gallery* $G = \{X_0, X_1, \ldots, X_n\}$ *that contains every chamber in* Σ *exactly once with* $X = X_0$*. This is possible as the apartment is thin and connected, so as a graph the apartment is connected and every chamber has degree two. There exists a simplicial map* $\omega_X : \Sigma \to \Sigma$*, such that* ω_X *fixes each vertex of the chamber* X *and* $\omega_X(X_i) = X$ *for each chamber in the gallery.*
For an object $y \in \Sigma$ *we know that* $\omega_X(y) \subset X$*. Define two elements* $y, z \in \Sigma$ *to be of the same type if and only if* $\omega_X(y) = \omega_X(z)$*.*
This definition of type is independent of the choice chamber X *and the choice of map* ω_X*.*

We see an explicit example of this in the next section.

Theorem 3.14. *Suppose* W *is the Weyl group of an apartment* Σ*,* X *a chamber in the apartment, with* S *the set of all reflections corresponding to the walls of* X*. Then:*

(i) W is generated by all the elements of S,

(ii) W is given by all type-preserving automorphisms of the apartment,

(iii) Given any two chambers X, Y in the apartment, there exists $g \in W$ such that $X^g = Y$, furthermore g is unique. Equivalently the Weyl group acts sharply transitively on the chambers of the apartment.

Another way a building is often defined is as in [13].

Remark 3.15. *Take a Coxeter system (W, S) and form a pair (Δ, δ). Δ is a non-empty set, and we call its elements chambers. $\delta : \Delta \times \Delta \to W$ such that, for all $X, Y \in \Delta$ the following holds:*

(i) $\delta(X, Y) = 1 \iff X = Y$,

(ii) Suppose $\delta(X, Y) = w$, and for $Z \in \Delta$ we have $\delta(Z, X) = s \in S$. Then $\delta(Z, Y) = sw$ or w,
Also if $\ell(sw) = \ell(w) + 1$ then $\delta(Z, Y) = sw$.

(iii) If $\delta(X, Y) = w$ then for each $s \in S$, there exists $Z \in \Delta$ satisfying $\delta(Z, X) = s$ and $\delta(Z, Y) = sw$.

We call X and Y s-adjacent if $\delta(X, Y) = s$. We write $X \sim_s Y$ if $X = Y$ or X and Y are s-adjacent.

3.2 A Useful Example

We will now track through the previous section with the example of the Fano plane. In places we do not verify the conditions of a definition or theorem for every case. Instead, for clarity and brevity we pick an illuminating example, provided the other cases follow similarly.

Consider the Fano plane as below.

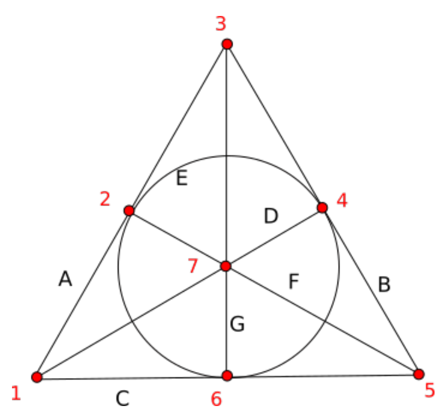


Figure 1: Labelling of the Fano Plane

Is this a complex? Let us check each condition of the definition.

(i) Define the elements of Δ to be given by all "singletons", that is lines and points. And "pairs" given by a point and a line with the condition that the point lies on the line. Note that for singletons we drop bracket notation, as a result, we use 1 in place of $\{1\}$. We take the partial order to be given by inclusion.
For example $1 \subset \{1, A\}$ and $1 \not\subset \{2, A\}$.

(ii) Given two elements of Δ we define their greatest lower bound to be their intersection as sets.
For example, $1 \cap \{1, A\} = 1$ and $1 \cap \{5, B\} = \emptyset$.

(iii) With the above definitions, for any $X \in \Delta$ we have that Δ_X is just another example of Example 3.1, as Δ_X can be taken to be the powerset of X. Hence this is a simplex.
For example, $\Delta_{\{1,A\}} = \{\emptyset, 1, A, \{1, A\}\}$.

An example of a subcomplex would be to take $\Gamma = \{\emptyset, 1, 2, 3, A, B, \{1, A\}, \{3, A\}, \{3, B\}\}$. This is a subset of Δ and also satisfies the containment condition. For example $\{3, B\} \in \Gamma$, and $3, B \in \Gamma$.

A point p gives $\Delta_p = \{\emptyset, p\}$ and a line L gives $\Delta_L = \{\emptyset, L\}$. By Definition 3.1 both these are rank 0 simplices, hence any singleton element is a rank 0 element.
For a pair $\{p, L\}$, we find $\Delta_{\{p,L\}} = \{\emptyset, p, L, \{p, L\}\}$, which is a rank 1 simplex. Therefore pairs are rank 1 elements, and as a result are the maximal elements.

We can see from the diagram of the Fano plane that every point p lies on at least some line L. And so both p and L are contained in the maximal element $\{p, L\}$. For that reason, this is a complex of rank one.
When are two maximal elements, $X \neq Y$, adjacent? From the definition in the previous section, we need to find z, a rank 0 element, such that $z \in X$ and $z \in Y$. We can see for our example this is equivalent to checking that $X \cap Y \neq \emptyset$. Here the notation $\cap$ is compatible with both the normal intersection of sets and also the notation of greatest lower bound as defined in the previous section.
For example $\{1, A\}$ is adjacent to both $\{1, B\}$ and $\{2, A\}$, whereas $\{1, B\}$ and $\{2, A\}$ are not adjacent.

Again from the diagram, we note that any two points are connected by a line, and any two lines share a common point. This means that for any pair of maximal elements we can find a chain of adjacent maximal elements connecting them. In particular, the maximum distance maximal elements can be apart is 3.
For example, for $\{1, A\}$ and $\{5, F\}$ we find the chain; $(\{1, A\}, \{1, C\}, \{5, C\}, \{5, F\})$.

In this example, every rank 0 element is contained in 3 maximal elements, as every line contains 3 points and every point lies on 3 lines.
For example 1 is contained in $\{1, A\}$, $\{1, C\}$ and $\{1, D\}$. And A is contained in $\{1, A\}, \{2, A\}$ and $\{3, A\}$.

Does Δ form a building?

(i) As we have verified above, Δ is thick.

(ii) In order to satisfy the second condition, it suffices to take $\mathcal{A}$ to be the set of all subcomplexes that are connected, thin and of rank 1.

(iii) Rather than verifying this condition for every pair of elements in Δ, let's pick a specific pair and verify it for this case. Other cases are similar.
Consider $\{1, A\}$ and $\{3, B\}$ in Δ. We can choose the subcomplex:
$\Sigma = \{\emptyset, 1, 3, 5, A, B, C, \{1, A\}, \{1, C\}, \{3, A\}, \{3, B\}, \{5, B\}, \{5, C\}\}$.
Indeed, $\{1, A\}, \{3, B\} \in \Sigma$. Σ is connected, visually this corresponds to taking the lines and vertices of the outer triangle in the Fano plane. This will give us a connected simplex. In addition it is thin as every line contains two points and so is contained in two maximal simplices given by taking the line itself and either of these two points. And similarly, each point lies on 2 lines. Finally, this is of rank 1, as its maximal elements are of rank 1.

(iv) Again let us just check this condition for the example of $\{1, A\}$ and $\{3, B\}$ in Δ with,

$$\Sigma = \{\emptyset, 1, 3, 5, A, B, C, \{1, A\}, \{1, C\}, \{3, A\}, \{3, B\}, \{5, B\}, \{5, C\}\} \quad \text{and,}$$
$$\Sigma' = \{\emptyset, 1, 3, 4, A, B, D, \{1, A\}, \{1, D\}, \{3, A\}, \{3, B\}, \{4, B\}, \{4, D\}\}.$$

First we need to find $\phi : \Sigma \to \Sigma'$ an isomorphism. The condition that subsets of our chosen points must be fixed means we need ϕ to satisfy $\phi(1) = 1, \phi(3) = 3$, $\phi(A) = A$ and $\phi(B) = B$. This gives very little freedom for the map, in fact now for ϕ to be an isomorphism we require that $\phi(5) = 4$ and $\phi(C) = D$. We can then define $\phi(\{x, X\}) = \{\phi(x), \phi(X)\}$. This maps all pairs into Σ', is well defined and makes the map a homomorphism. Resultantly σ is the required map.

We now call $\Sigma = \{\emptyset, 1, 3, 5, A, B, C, \{1, A\}, \{1, C\}, \{3, A\}, \{3, B\}, \{5, B\}, \{5, C\}\}$ an apartment. The maximal elements, in this case pairs, we now call chambers. And the elements that are of rank one lower than maximal, in this cases points or lines, we call walls.

Can we show that two apartments we choose are isomorphic? Consider the two apartments:

$$\Sigma = \{\emptyset, 1, 3, 5, A, B, C, \{1, A\}, \{1, C\}, \{3, A\}, \{3, B\}, \{5, B\}, \{5, C\}\},$$

$$\Pi = \{\emptyset, 2, 4, 7, E, F, D, \{2, E\}, \{2, F\}, \{4, E\}, \{4, D\}, \{7, D\}, \{7, F\}\}.$$

Then we can take the isomorphism between them defined by:

$$f(1) = 2, \quad f(3) = 4 \quad \text{and} \quad f(5) = 7.$$

$$f(A) = E, \quad f(C) = F \quad \text{and} \quad f(B) = D$$

$$f(\{1, A\}) = \{2, E\}, \quad f(\{1, C\}) = \{2, F\}, \quad f(\{3, A\}) = \{4, E\}.$$

$$f(\{3, B\}) = \{4, D\}, \quad f(\{5, B\}) = \{7, D\}, \quad f(\{5, C\}) = \{7, F\}.$$

And so these two apartments are isomorphic, we have also seen a case above showing that two apartments containing a common point are isomorphic.

Recall that before we found the chain $\{\{1, A\}, \{1, C\}, \{5, C\}, \{5, F\}\}$ between $\{1, A\}$ and $\{5, F\}$. We now call this a gallery of length three connecting $\{1, A\}$ and $\{5, F\}$.
We see from the diagram of the Fano plane that this is the shortest such gallery, and clearly, it does not stammer as we have no repeated chambers. Therefore $dist(\{1, A\}, \{5, F\}) = 3$.

We know each apartment is a thin complex of rank d, in this case $d = 1$. And so we can find a folding of our apartment $\Sigma = \{\emptyset, 1, 3, 5, A, B, C, \{1, A\}, \{1, C\}, \{3, A\}, \{3, B\}, \{5, B\}, \{5, C\}\}$.
Define the folding $\sigma_a : \Sigma \to \Sigma$ by:

$$\sigma_x(1) = 1, \quad \sigma_x(5) = 1 \quad \text{and} \quad \sigma_x(3) = 3.$$

$$\sigma_x(A) = A, \quad \sigma_x(B) = A \quad \text{and} \quad \sigma_x(C) = C.$$

$$\sigma_x(\{1, A\}) = \{1, A\} = \sigma_x(\{5, B\}).$$

$$\sigma_x(\{3, A\}) = \{3, A\} = \sigma_x(\{3, B\}).$$

$$\sigma_x(\{1, C\}) = \{1, C\} = \sigma_x(\{5, C\}).$$

Clearly $\sigma_x^2 = \sigma_x$. For example, $\sigma_x^2(\{5, B\}) = \sigma_x(\{1, A\}) = \{1, A\} = \sigma_x(\{5, B\})$.
The chambers in $\sigma_x(\Sigma)$ are $\{1, A\}$, $\{3, A\}$ and $\{1, C\}$. As a result of the way we have defined the map, each chamber has 2 preimages. Thus this is indeed a folding.

Let us verify Lemma 3.10 for an example, and use it to find a reflection in a wall.
We take the two adjacent chambers $X = \{1, C\}$ and $Y = \{5, C\}$. Let σ_x be as above. And define $\sigma_y : \Sigma \to \Sigma$ as follows.

$$\sigma_y(1) = 5, \quad \sigma_y(5) = 5 \quad \text{and} \quad \sigma_y(3) = 3.$$

$$\sigma_y(A) = B, \quad \sigma_y(B) = B \quad \text{and} \quad \sigma_y(C) = C.$$

$$\sigma_y(\{1, A\}) = \{5, B\} = \sigma_y(\{5, B\}).$$

$$\sigma_y(\{3, A\}) = \{3, B\} = \sigma_y(\{3, B\}).$$

$$\sigma_y(\{1, C\}) = \{5, C\} = \sigma_y(\{5, C\}).$$

By the same reasoning that showed σ_x is a folding, we can show σ_y is also a folding of the apartment.

We can check these two foldings satisfy the conditions of Lemma 3.10:

(i) $\sigma_x(X) = \sigma_x(\{1, C\}) = \{1, C\} = X$,
$\sigma_y(Y) = \sigma_y(\{5, C\}) = \{5, C\} = Y$.

(ii) $\sigma_x(\Sigma) = \{1, 3, A, C, \{1, A\}, \{3, A\}, \{1, C\}\}$,
$\sigma_y(\Sigma) = \{3, 5, B, C, \{5, B\}, \{3, B\}, \{5, C\}\}$
Therefore $\sigma_x(\Sigma) \cup \sigma_y(\Sigma) = \Sigma$.

(iii) $\sigma_x(\Sigma) \cap \sigma_x(\Sigma)$ is given by the two singletons 3 and C. In particular, $\sigma_x(\Sigma) \cap \sigma_x(\Sigma)$ contains no chambers.

(iv) Define θ by:

$$\theta(Z) = \begin{cases} \sigma_x(Z) & \text{if } Z \in \sigma_y(\Sigma), \\ \sigma_y(Z) & \text{if } Z \in \sigma_x(\Sigma). \end{cases}$$

We get the following map:

$$\theta(1) = 5, \quad \theta(3) = 3 \quad \text{and} \quad \theta(5) = 1.$$

$$\theta(A) = B, \quad \theta(B) = A \quad \text{and} \quad \theta(C) = .C$$

$$\theta(\{1, A\}) = \{5, B\}, \quad \theta(\{3, A\}) = \{3, B\}, \quad \theta(\{1, C\}) = \{5, C\}.$$

$$\theta(\{5, B\}) = \{1, A\}, \quad \theta(\{3, B\}) = \{3, A\}, \quad \theta(\{5, C\}) = \{1, C\}.$$

Clearly we have that $\theta(\{p, L\}) = \{\theta(p), \theta(L)\}$ for each p a point and L a line, therefore θ is an automorphism. This can be represented as below.

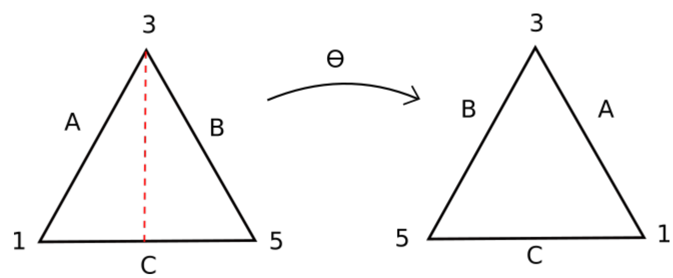


Figure 2: Reflection in the wall C

We consider another pair of adjacent chambers within the apartment, $X' = \{3, B\}$ and $Y' = \{5, B\}$, and find the reflection in their common wall B. We define $\sigma_{x'}$ and $\sigma_{y'}$ as follows.

$$\sigma_{x'}(1) = 1, \quad \sigma_{x'}(3) = 3 \quad \text{and} \quad \sigma_{x'}(5) = 3.$$

$$\sigma_{x'}(A) = A, \quad \sigma_{x'}(B) = B \quad \text{and} \quad \sigma_{x'}(C) = A.$$

$$\sigma_{x'}(\{1, A\}) = \{1, A\} = \sigma_{x'}(\{1, C\}).$$

$$\sigma_{x'}(\{3, A\}) = \{3, A\} = \sigma_{x'}(\{5, C\}).$$

$$\sigma_{x'}(\{3, B\}) = \{3, B\} = \sigma_{x'}(\{5, B\}).$$

$$\sigma_{y'}(1) = 1, \quad \sigma_{y'}(3) = 5, \quad \text{and} \quad \sigma_{y'}(5) = 5.$$

$$\sigma_{y'}(A) = C, \quad \sigma_{y'}(B) = B \quad \text{and} \quad \sigma_{y'}(C) = C.$$

$$\sigma_{y'}(\{1, A\}) = \{1, C\} = \sigma_{y'}(\{1, C\}).$$

$$\sigma_{y'}(\{3, A\}) = \{5, C\} = \sigma_{y'}(\{5, C\}).$$

$$\sigma_{y'}(\{3, B\}) = \{5, B\} = \sigma_{y'}(\{5, B\}).$$

With this we get the map θ' given as:

$$\theta'(1) = 1, \quad \theta'(3) = 5 \quad \text{and} \quad \theta'(5) = 3.$$

$$\theta'(A) = C, \quad \theta'(B) = B \quad \text{and} \quad \theta'(C) = A.$$

$$\theta'(\{1, A\}) = \{1, C\}, \quad \theta'(\{3, A\}) = \{5, C\} \quad \text{and} \quad \theta'(\{3, B\}) = \{5, B\}.$$

$$\theta'(\{1, C\}) = \{1, A\}, \quad \theta'(\{5, C\}) = \{3, A\} \quad \text{and} \quad \theta'(\{5, B\}) = \{3, B\}.$$

This is another order 2 automorphism, which can be represented as below.

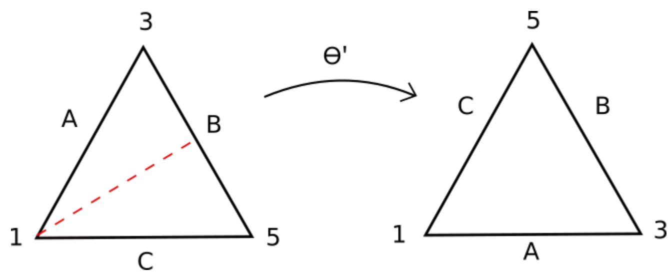


Figure 3: Reflection in the wall B

$\theta \circ \theta'$ is then also an automorphism. In particular, as it is a homomorphism it suffices to define its effect on the points and lines.

$$\theta \circ \theta'(1) = 5, \quad \theta \circ \theta'(3) = 1 \quad \text{and} \quad \theta \circ \theta'(5) = 3.$$

$$\theta \circ \theta'(A) = C, \quad \theta \circ \theta'(B) = A \quad \text{and} \quad \theta \circ \theta'(C) = B.$$

This can be represented as below.

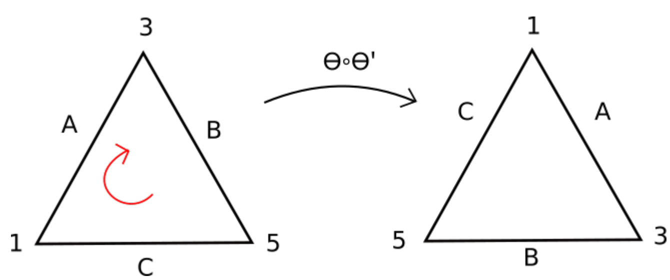


Figure 4: $\theta \circ \theta'$

Looking at these diagrams we can see with θ and θ', we generate the symmetry group of the equilateral triangle, which is isomorphic to $Sym(3)$. In later sections we show that the Weyl group is indeed $Sym(3)$.

What are the types of objects of our building?
Consider the apartment:

$$\Sigma = \{\emptyset, 1, 3, 5, A, B, C, \{1, A\}, \{1, C\}, \{3, A\}, \{3, B\}, \{5, B\}, \{5, C\}\}.$$

Let us fix the chamber $\{1, A\}$, and form the gallery:

$$G = (\{1, A\}, \{1, C\}, \{5, C\}, \{5, B\}, \{3, B\}, \{3, A\}).$$

Consider the simplicial map $\omega : \Sigma \to \Sigma$ that fixes the vertices of $\{1, A\}$ and maps all chambers of the gallery to $\{1, A\}$.
Hence we know that $\omega(1) = 1$ and $\omega(A) = A$.
$\omega(\{1, C\}) = \{1, A\}$, using the fact that $\omega(1) = 1$ we deduce that $\omega(C) = A$.
$\omega(\{5, C\}) = \{1, A\}$ with $\omega(C) = A$ and so $\omega(5) = 1$.
By iterating this process on all chambers of the gallery we find that:

$$\omega(1) = \omega(3) = \omega(5) = 1, \qquad \omega(A) = \omega(B) = \omega(C) = A.$$

Each element, x, of our building is contained in some chamber X. We know that any two chambers in a building are contained in some apartment. And so let us take the two chambers $\{1, A\}$ and X. We can repeat the process above to find the type of our element x. For example suppose we want to find the type of D. D is contained in the chamber $\{5, D\}$, with $\{1, A\}$ and $\{5, D\}$ both contained in the apartment below.

$$\Sigma' = \{\emptyset, 1, 3, 4, A, B, D, \{1, A\}, \{1, D\}, \{3, A\}, \{3, B\}, \{4, B\}, \{5, D\}\}$$

Replicating the process above with ω', we find $\omega'(D) = A$, we also get that $\omega'(4) = 1$.

Carrying this out for the remaining elements of our building, we find that our objects split into two types. Type $1 = \{1, 2, 3, 4, 5, 6, 7\}$ and type $2 = \{A, B, C, D, E, F, G\}$.

4 (B, N)-pairs

We now look at (B, N)-pairs. These are of interest to us because, as we shall see each (B, N)-pair corresponds to exactly one building. In addition, each building corresponds to any number of (B, N)-pairs [8], we will see one such correspondence. They also give an insight into why there are differences between chamber graphs of buildings and chamber graphs of geometries which we look at in Section 5.

Definition 4.1. *Given a group G and two subgroups B, N, these form a (B, N) pair if the following hold:*

(i) $G = \langle B, N \rangle$,

(ii) $H = B \cap N \trianglelefteq N$,

(iii) $W = N/H$ *has a generating set given by* $S = \{w_i\}_{\{i \in I\}}$*, such that* $w_i^2 = 1$ *for all* $i \in I$. *Let* $\pi : N \to W$ *be the natural homomorphism,*

(iv) If $\pi(n_i) = w_i$ *for some* $n_i \in N$*, then* $n_i B n_i = n_i^{-1} B n_i \neq B$,

(v) If $w_i \in S$ *and* $w \in W$ *then* $w_i B w \subseteq B w_i w B \cup B w B$.

B is called the Borel Subgroup, this is unique up to conjugation. H is called the Cartan Subgroup, and W the Weyl group of G.
(W, S) is a Coxeter system, and the number of generators in S is called the rank.
We call subgroups containing any conjugate of B in G parabolic subgroups of G. A subgroup containing a specific choice of conjugate of B is called a standard parabolic subgroup.

The following Theorem gives an example of a (B, N)-pair will be useful in Section 6.

Theorem 4.2. *For $GL_n(k)$, where k is a field, the following subgroups form a (B, N)-pair:*

1. *$B = \{A \in G \mid a_{i,j} = 0$ for $i > j\}$, the upper triangular matrices,*
2. *$N = \{A \in G \mid A$ has exactly one non-zero entry in each row and column$\}$, the monomial or generalised permutation matrices.*

This proof is from [27].

Proof. We verify the five conditions in the definition 4.1.
(ii) $H = B \cap N$ is the set of diagonal matrices. These are indeed normal in N.

(iii) For each $n \in N$ we can write $n = ph$ where p is a permutation matrix and $h \in H$ is a diagonal matrix.
A permutation matrix just permutes the basis $\{e_1, \dots, e_n\}$. Therefore P, the group of permutation matrices, is a subgroup of N.
Thus $N = PH$ with $P \cap H = \{I_n\}$.
$P = \langle n_i \mid i = 1, \dots, n-1\rangle$ where n_i is the identity matrix with e_i and e_{i+1} interchanged.

Remark 4.3. *In this case P is the Weyl group of $GL_n(k)$.*

(i) Let $x \in G$, then there exists $b_1, b_2 \in B$ such that $b_1 x b_2 \in N$. This follows by applying suitable row and column operations to x.
Therefore $x \in b_1^{-1} N b_2^{-1} \subseteq BNB$, giving $G \subseteq BNB$. Also $B, N \leq G$, implies that $BNB \subset G$.
Ergo $G = \langle B, N\rangle$, in particular $G = BNB$.

(iv) If we take $b = I_n + e_i e_{i+1}^T \in B$ then $n_i b n_i = I_n + e_{i+1} e_i^T \notin B$.

(v) We define various subgroups of B.
$U = \{A \in G \mid a_{ij} = 0 \text{ for } i > j, a_{ii} = 1\}$, uni-upper triangular matrices,
$U_i = \{A \in U \mid a_{i,i+1} = 0\}$,
$X_{ij} = \{I_n + \lambda e_j e_i^T \mid \lambda \in GF(q)\}$,
$X_i = X_{i,i+1}$ and $X_{-i} = X_{i+1,i}$.

Facts:

1. $U \trianglelefteq B$,
2. $B = UH = HU$,
3. $U = X_i U_i = U_i X_i$,

4. $n_iX_in_i = X_{-i}$ and $n_iU_in_i = U_i$.

Consider n_iBn where $n \in N$,

$$\begin{aligned} n_iBn &= n_iHUn \quad \text{by } 2 \\ &= Hn_iUn \quad \text{as } H \trianglelefteq B \\ &= Hn_iU_iX_in \quad \text{by } 3 \\ &= HU_in_iX_in \quad \text{by } 4 \\ &\subseteq Bn_iX_in \quad \text{as } HU_i \leq B \end{aligned}$$

And so $n_iBn \subseteq Bn_iX_in$. $(*)$

Let $n \in N$ correspond to $\sigma \in Sym(n)$. In other words n is the generalized permutation matrix with its non zero entries in positions $(i, (i)\sigma)$ for $i = 1 : n$.
And so $n^{-1}X_{i,j}n = X_{(i)\sigma,(j)\sigma}$
$n_iX_in = n_inn^{-1}X_in = n_inX_{(i)\sigma,(j)\sigma}$.
From this we get that there are two possibilities:

1. $(i)\sigma < (i+1)\sigma$,
2. $(i)\sigma > (i+1)\sigma$.

<u>Case 1</u>
$(i)\sigma < (i+1)\sigma$ implies $X_{(i)\sigma,(i+1)\sigma} \leq B$.
So $n_iXn = n_inX_{(i)\sigma,(i+1)\sigma} \subseteq n_inB$ $(**)$

$$\begin{aligned} n_iBn &\subseteq Bn_iXn \quad \text{by } * \\ &\subseteq Bn_inB \quad \text{by } ** \\ &\subseteq Bn_inB \cup BnB \end{aligned}$$

It follows that (v) holds for Case 1.

<u>Case 2</u>
Set $n' = n_in$. And so $n = n_in_in = n_in'$.

$i \xrightarrow{n_i} i+1 \xrightarrow{n} (i+1)n$ implies $(i)n' = (i+1)n$. And $i+1 \xrightarrow{n_i} i \xrightarrow{n} (i)n$ implies $(i+1)n' = (i)n$.
This means that $(i)n' < (i+1)n'$.
$n_iX_in = n_iX_in_in' = X_{-i}n'$, with the second equality holding by (4).
And so $n_iX_in = X_{-i}n'$. $(\dagger)$

We use $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to represent the matrix A that is the identity but with
$A(i : i+1, i : i+1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

For any $\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix} \in X_{-i} \neq id$ (and so $\lambda \neq 0$) we have,

$$\begin{aligned}
\begin{pmatrix}1 & 0\\ \lambda & 1\end{pmatrix} &= \begin{pmatrix}1 & \lambda^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}-\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & \lambda^{-1}\\ 0 & 1\end{pmatrix}\\
&\in UHn_iU\\
&= Bn_iU \qquad \text{by } 2\\
&\subseteq Bn_iB\\
\Rightarrow X_{-i} &\subseteq B \cup Bn_iB\\
\Rightarrow X_{-i}n' &\subseteq Bn' \cup Bn_iBn'\\
\Rightarrow X_{-i}n' &\subseteq Bn' \cup Bn_in'B \qquad \text{by case 1 as } (i)n' < (i+1)n.
\end{aligned}$$

And so $X_{-i}n' \subseteq Bn' \cup Bn_in'B.$ $\qquad (\dagger\dagger)$

$$\begin{aligned}
n_iBn &\subset Bn_iX_in \qquad \text{by } *\\
&= BX_in' \qquad \text{by } \dagger\\
&\subseteq Bn' \cup Bn_in'B \qquad \text{by } \dagger\dagger\\
&= Bn_in \cup BnB \qquad \text{as } n = n_in'\\
&\subseteq Bn_inB \cup BnB.
\end{aligned}$$

Hence for Case 2 (v) holds.
This means (v) holds in all cases.

We have checked all conditions in 4.1, accordingly B and N form a (B, N)-pair for $GL_n(k)$.
□

Remark 4.4. *As discussed in Remark 4.3, P is the Weyl group of $GL_n(k)$. Hence the Weyl group of $GL_n(q)$ is isomorphic to Sym(n) by taking the isomorphism defined on generators by; $n_i \mapsto (i, i+1)$ for $i = 1, \ldots, n-1$.*

As we will see in Section 6.2, the Weyl group controls a lot of the behaviour of the chamber graph of a building, which we define in Section 5. For groups that do not have Weyl groups, such as the Sporadics, we can also calculate chamber graphs of their associated geometries. Because these groups do not have a Weyl group we would not necessarily expect these chamber graphs to look like those of buildings. However, in some cases they get very close.

We would like to give examples of the correspondence between buildings and (B, N)-pairs. In order to do this we require the following theorem, definition and remark.

Theorem 4.5. *Suppose G is a group with a (B, N)-pair then:*

1. *$G = BNB$,*
2. *$J \subseteq I$, $J \neq \emptyset$, we can set $W_J = \langle w_i \mid i \in J\rangle$. With this we have that $BN_JB \leq G$, where $N_J/H = W_J$.*

Proof. First we prove (2) by applying the subgroup criterion.
As J is non empty, clearly we have that BN_JB is non empty.
We also have:

$$(BN_JB)^{-1} = B^{-1}N_J^{-1}B^{-1} = BN_JB,$$

with the second equability holding because both B and N_J are subgroups.
Thus BN_JB is closed under inversion.

Suppose that $n \in N_J$, and so $\pi(n) = w_{i_1} \dots w_{i_k}$ with $i_1, \dots, i_k \in J \subseteq I$ by 4.1(iii).
If $n_{i_j} \in N$ such that $\pi(n_{i_j}) = w_{i_j}$ then $n \in n_{i_1} \dots n_{i_k}H$.
Hence $nBN_JB \subseteq n_{i_1} \dots n_{i_k}HBN_JB$
$\subseteq n_{i_1} \dots n_{i_k}BN_JB,$ as H is a subset of B.
By 4.1 (v) $n_{i_k}BN_JB \subseteq Bn_{i_k}N_JB \cup BN_JB$.
$Bn_{i_k}N_JB \cup BN_JB \subseteq BN_JB \cup BN_JB = BN_JB$, as N_J is a subgroup containing n_{i_k}.
By running through the n_{i_k}s we find the following.

$$\begin{aligned} nBN_JB &\subseteq n_{i_1} \dots n_{i_{k-1}}(n_{i_k}BN_JB) \\ &\subseteq n_{i_1} \dots n_{i_{k-1}}BN_JB \\ &\subseteq \dots \\ &\subseteq BN_JB. \end{aligned}$$

Hence $N_JBN_JB \subseteq BN_JB$.
Multiplying by B we find, $(BN_JB)(BN_JB) = BN_JBN_JB \subseteq BN_JB$.
Therefore BN_JB is closed under multiplication.
Resultantly $BN_JB \leq G$, proving (2).

For (1) take $J = I$ and so $N_I = N$. Applying (1) gives $BNB \leq N$.
And now since $\langle B, N\rangle = G$ by 4.1 (i), we have that $BNB = G$. □

Definition 4.6. *For $J \subseteq I$ we set $P_J = BN_JB$.*

Remark 4.7. *If we have a group M such that $G \geq M \geq B$ then $M = P_J$ for some $J \subseteq I$. And so these P_J form the standard parabolic subgroups.*

The construction for a building from a (B, N)-pair is given below, details for this are from [12].

Example 4.8. *Suppose we have a group G with a (B, N)-pair. We can take all proper parabolic subgroups and form a partial order. We do this by taking the face inclusion to be reverse inclusion in G. This then gives a building.*
The apartment containing B is given by:

$$\mathcal{A} = \{nPn^{-1} \mid P \textit{ a standard parabolic subgroup and } n \in N\}.$$

Now for any $g \in G$ we have that,

$$g\mathcal{A} = \{gnPn^{-1}g^{-1} \mid P \textit{ a standard parabolic subgroup and } n \in N\}$$

is also an apartment.
The action of G on the set parabolic subgroups is given by conjugation.

Going the other way, if we are given a building we are able to find a (B, N)-pair, the construction for which is given below and comes from [18].

Example 4.9. *Suppose we have a Building Δ, containing an apartment Γ, which in turn contains a chamber γ_0. Let G to be the group of all the type preserving automorphisms on the building. Suppose that for any other apartment Π, with a chamber $\gamma' \in \Pi$ we have that $\sigma(\gamma_0) = \gamma'$, and $\sigma(\Gamma) = \Pi$ for some $\sigma \in G$. Set $B = Stab_G(\gamma_0)$ and $N = Stab_G(\Gamma)$. These subgroups of G form a (B, N)-pair. The set S, that generates the Weyl group, is given by all the reflections of the walls of γ_0. We can also recover H, that Cartan subgroup, but taking the subgroup formed by the elements of G that fix Γ point-wise.*

Another way to express part one of Theorem 4.5 is by making a slight abuse of notation, we directly identify $w_i = \pi(n_i)$ with n_i.

Theorem 4.10. *For G a group with a (B, N)-pair we have that:*

$$G = BWB = \dot{\bigcup_{w \in W}} BwB.$$

For each $w \in W$, we call BwB a Bruhat cell.

We will see that when we start to form chamber graphs in later sections, it proves useful to identify B-orbits with the double cosets of B in G. That is B^x identified with $BxB = \{b_1 x b_2 \mid b_i \in B\}$. By the Bruhat decomposition, we only need to run over entries in the Weyl group to find all double coset representatives. This is one of the reasons Weyl groups are so important. We will make use of this comment later.
In order to make this identification we need the following theorem which we will make use of in Section 5.

Theorem 4.11. *The Borel subgroup, B, of a group G with a (B, N)-pair is self-normalizing.*

Proof. Suppose $x \in G$ and $x \notin B$.
By Theorem 4.10 we know that $x = b_1 w b_2$, with $b_1, b_2 \in B$ and $w \in W$.
As $x \notin B$ we know that $w \neq id$.

$$\begin{aligned} \text{And so } x^{-1}Bx &= (b_1 w b_2)^{-1} B b_1 w b_2 \\ &= b_2^{-1} w^{-1} b_1^{-1} B b_1 w b_2 \\ &= b_2^{-1} w^{-1} B w b_2 \quad \text{As } B \leq N_G(B) \\ &\neq b_2^{-1} B b_2 \quad \text{By 4.1 (iv)} \\ &= B. \end{aligned}$$

This means that for $x \notin B$ we have $x \notin N_G(B)$, and so $N_G(B) \leq B$.
For any group $B \leq N_G(B)$ and so $N_G(B) = B$.
Accordingly B is self-normalizing. □

5 Chamber Graphs

In this section, we see how to form chamber graphs from group geometries. For buildings, we can form group geometries such that the corresponding chamber graph provides another way to view the building. This correspondence was fist developed by Tits in [31]. In this case, we see certain properties arising, as we discuss in the second part of this section. When are group geometries do not come from a building we can still form their chamber graphs, it is then interesting to investigate how much "building-like" behaviour these chamber graphs exhibit.

In order to define chamber graphs we first need to define geometries.

Definition 5.1. *A geometry over a set I is a triple $(\Gamma, \tau, *)$ where*

(i) *Γ is a non empty set,*

(ii) *$\tau : \Gamma \to I$ is a surjective map,*

(iii) *$*$ is a symmetric relation on Γ such that, for all $a, b \in \Gamma$, $a * b$ implies $\tau(a) \neq \tau(b)$.*

We say $a \in \Gamma$ has type i if $\tau(a) = i$. The rank of the geometry is $rank(\Gamma) = |I|$.
We can define the two sets $\Gamma_i = \{a \in \Gamma \mid \tau(a) = i\}$ and $\Gamma_a = \{b \in \Gamma \mid a * b\}$.
A flag is a set of pairwise incident elements (with respect to $*$) of Γ. The rank of the flag is $|\tau(F)|$, where $\tau(F) = \{\tau(a) \mid a \in F\}$. A flag is maximal if $|\tau(F)| = rank(\Gamma)$, we denote the set of all maximal flags by $\mathcal{C}(\Gamma)$ or just $\mathcal{C}$.

Given a geometry Γ, we can construct the chamber system for Γ as follows:

1. Let $\mathcal{C}$ be the set of all maximal flags of Γ,

2. $F_1, F_2 \in \mathcal{C}$ with $F_1 \neq F_2$, are i-adjacent if and only if $\tau(F_1 \cap F_2) = I \backslash \{i\}$.

Definition 5.2. *Given a chamber system as defined above, the chamber graph is formed by taking vertices to be elements of $\mathcal{C}$. We now call these vertices chambers. Two chambers are adjacent in the graphical sense if they are adjacent in the sense defined above.*

Fix a chamber γ_0 of $\mathcal{C}$ and set $B = Stab_G(\gamma_0)$ as in Example 4.9.
Given $\gamma_0, \gamma_1 \in \mathcal{C}$, we define a geodesic between them to be any path of shortest distance joining γ_0 and γ_1.
The distance between any two chambers, $\gamma_0, \gamma_1 \in \mathcal{C}$, is taken to be the number of edges in any geodesic between them, we denote this by $d(\gamma_0, \gamma_1)$.
The i^{th} disc of γ_0 is defined to be $\Delta_i(\gamma_0) = \{\gamma \in C \mid d(\gamma_0, \gamma) = i\}$.
The geodesic closure of $D \subseteq \mathcal{C}$, denoted $\overline{D}$, is the superset of D consisting of all chambers lying on any geodesic between any two chambers in D.
Two chambers are opposite if they are at maximal distance to one another, equivalently their distance is equal to the diameter of the chamber graph.
We define maximal opposite sets to be subsets of $\mathcal{C}$ of maximal size such that each pair of chambers in the set are opposite.

We would like a way to form geometries and chamber graphs from groups, to do this we need the following definitions.

Definition 5.3. *H is a maximal subgroup of a group G if:*

(i) $H \neq G$,

(ii) $H \leqslant K \leqslant G$ *implies* $K = G$ *or* $K = H$.

Example 5.4. *If G is a finite group and H is a subgroup of G of prime index, then H is a maximal subgroup.*

Definition 5.5. *Suppose G is a group with $|G| = a \cdot p^k$, and p prime such that $p \nmid a$. We call a subgroup H a Sylow subgroup of G if $|H| = p^k$.*

Definition 5.6. *Let G be a finite group and p some prime dividing the order of G. Let S be a Sylow p-subgroup of G and $B = N_G(S)$. A subgroup P, with $B \leqslant P \leqslant G$ such that B is contained in exactly one maximal subgroup of P, is called a minimal parabolic subgroup of G with respect to B.*

Definition 5.7. *Let G and B be as above, $M = \{P_1, \ldots, P_n\}$ is a minimal parabolic system of G, of rank n and characteristic p, if:*

(i) each P_i is a minimal parabolic subgroup,

(ii) $G = \langle M \rangle$,

(iii) G cannot be generated by any proper subset of M.

We will see examples of minimal parabolic subgroups and systems in section 6.1.

The next example is from [24], and will prove helpful in later sections.

Example 5.8. *Let G be a group, S is a Sylow p subgroup and $B = N_G(S)$. Let $M = \{P_1, \ldots, P_n\}$ be a minimal parabolic system of G with respect to B. We can form a chamber system by taking the set of chambers to be the conjugates of B, with B^g i-adjacent to B^h if and only if $gh^{-1} \in P_i$.*

When G is of Lie Type and p is the characteristic of the field with Lie rank ≥ 2, there is only one such chamber system, which is the chamber system corresponding to the building of G. When forming the chamber graph of a building we preserve all the important information of the building. This is discussed further in the next subsection and we will see an example of this in terms of the apartments at the very in Section 6.1.
If G is not of Lie Type then we can still form the chamber graph of a group geometry, we do this in Section 8, but this will not necessarily be a building. We will explore this more in later sections.

Remark 5.9. *The name Group of Lie Type comes from the fact that a lot of these groups are associated with Lie Algebras. We do not define groups of Lie type here, it suffices to know that both $GL_3(2)$ and $GL_4(2)$ are of Lie type.*

Remark 5.10. *It is easier for us to work with cosets of B rather than conjugates of B so we would like to rephrase Example 5.8 in terms of cosets.*
In the examples we see in this project we can take the set of chambers to be the set of cosets of B in G, we say that Bg and Bh are i adjacent ($Bg \sim_i Bh$) if and only if $gh^{-1} \in P_i$.

Proof. We first need to check that this relation is independent of our choice of representative.
Let $Bg \sim_i Bh$.
Suppose $Bg = Bg'$ and $Bh = Bh'$. We then find that $g' = b_1 g$ and $h' = b_2 h$ for $b_1, b_2 \in B$.
Using these equalities, $g'h'^{-1} = b_1 g(b_2 h)^{-1} = b_1 g h^{-1} b_2$.
We know $gh^{-1} \in P_i$, and $b_1, b_2 \in B \leq P_i$, hence $g'h'^{-1} = b_1 g h^{-1} b_2 \in P_i$.
Thus $Bg' \sim_i Bh'$.

We also need to check this gives the same relation as before.
We do this by forming a bijection between the set of cosets and the set of conjugates by sending Bg to $g^{-1}Bg$.
As before we are using coset representatives to define the map and so we need to check that this is well defined.
Suppose $Bh = Bk$, meaning $h = bk$ for some $b \in B$.

$$\begin{aligned} h^{-1}Bh &= (bk)^{-1}B(bk) \\ &= k^{-1}b^{-1}Bbk \\ &= k^{-1}Bk \end{aligned}$$

This proves that the map is well defined.

In all the cases we see in Section 6, we know that B is self-normalizing by Theorem 4.11, and so $N_G(B) = B$. By [15] we know that the number of cosets of $N_G(B)$ in G equals the number of conjugates of B in G. As $N_G(B) = B$ we find that $|\{Bh \mid h \in G\}| = |\{h^{-1}Bh \mid h \in G\}|$.
This means that to show the map is bijective it suffices to show it is injective.
$h^{-1}Bh = k^{-1}Bk$ if and only if $(hk^{-1})^{-1}Bhk^{-1} = kh^{-1}Bhk^{-1} = B$.
Equivalently $hk^{-1} \in N_G(B) = B$.
$hk^{-1} \in B$ is precisely the same as requiring $Bh = Bk$.
On that account $h^{-1}Bh = k^{-1}Bk$ if and only if $Bh = Bk$.
Therefore the map is an injection. □

Remark 5.11. *The B-orbits are given by $\{Bxy \mid y \in B\} = BxB$, this is exactly a double cosets of B in G. Hence we also identify B-orbits with double cosets.*

We make use of this Remark in the proof of the following theorem.

Theorem 5.12. *Using the construction as in Remark 5.8 we find that each disc of the chamber graph is a union of B-orbits. Furthermore, for two B-orbits, B_1 and B_2, if there is a chamber in B_1 that is i-adjacent to a chamber in B_2 then every chamber in B_1 is i-adjacent to some chamber in B_2.*

Proof. All the groups we encounter are finite and so $[G : B]$ is finite. As a result, there are finitely many chambers and the number of discs must be finite. Therefore we can proceed by induction on the on the number of discs.

Firstly we show that disc one is a union of B-orbits.
If $Bg \in \Delta_1(\gamma_0)$ then $Bg \sim_i B$ for some i, and so $g = g1^{-1} \in P_i$.
Let Bh be another chamber in the orbit of Bg. Identifying orbits with double cosets gives $BhB = BgB$. Hence $h = b_1 g b_2$ for some $b_1, b_2 \in B$.
B is a subgroup of P_i, thus $b_1 P_i b_2 = P_i$. Therefore $h1^{-1} = h = b_1 g b_2$ lies in P_i.
Resultantly $B \sim_i Bh$. The result hold for the first disc.

Assume the result holds up to disc j.

Consider an orbit in the j^{th} disc, containing a chamber Bx. Suppose $Bx \sim_i Bg$, where $Bg \in \Delta_{j+1}(\gamma_0)$.
Let Bh be another chamber in the same double coset containing Bg. This cannot have appeared in any of the previous j by our inductive step.
We need to show that there exists some chamber By in the orbit BxB such that $By \sim_i Bh$.
Again we have that: $BxB = ByB$ and that $BgB = BhB$. Therefore $x = b_1 y b_2$ and $g = b_3 h b_4$ for some $b_1, b_2, b_3, b_4 \in B$.
$Bx \sim_i Bg$ implies $xg^{-1} \in P_i$.
By in equalities above $xb_4^{-1}h^{-1}b_3^{-1} \in P_i$.
As B is a subgroup of P_i, we find $xb_4^{-1}h^{-1} \in P_i$.
Hence let us choose $y = xb_4^{-1}$ or equivalently $b_1 = id$, and $b_2 = b_4$.
This gives $By \sim_i Bh$.
And hence we have shown $\Delta_{j+1}(\gamma_0)$ is a union of B-orbits.
The comment on adjacency follows by the same argument. □

We know that for our chamber graph construction, G is generated by the P_is. Because of this, in every case we see the chamber graphs will be connected.

Theorem 5.13. *$\mathcal{C}$ is connected if and only if $G = \langle P_i \mid i \in I \rangle$.*

Proof. First let us assume that $G = \langle P_i \mid i \in I \rangle$ and show that $\mathcal{C}$ is connected.
It is clear that for $p \in P_i$ we have $B \sim_i Bp$ because $p^{-1} \in P_i$.
Similarly if $p, q \in P_i$ then $Bp \sim_i Bq$ as $pq^{-1} \in P_i$.
To show that $\mathcal{C}$ is connected, it suffices to check that every chamber is connected to B by some sequence of chambers.
Consider an arbitrary $x \in G = \langle P_i \mid i \in I \rangle$
Hence $x = p_1 p_2 \dots p_m$ such that for each $1 \leq j \leq m$, $p_j \in P_i$ for some $i \in I$.
$B \sim Bq_m$ by our earlier comment.
$Bq_m \sim Bq_{m-1}q_m$ as $q_m(q_{m-1}q_m)^{-1} = q_m q_m^{-1} q_{m-1}^{-1} = q_{m-1}^{-1} \in P_i$ for some $i \in I$.
$Bq_{m-1}q_m \sim Bq_{m-2}q_{m-1}q_m$ as $q_{m-1}q_m(q_{m-2}q_{m-1}q_m)^{-1} = q_{m-2}^{-1} \in P_i$ for some $i \in I$.
$\vdots$
$Bq_2 \dots q_m \sim Bq_1 q_2 \dots q_m$ as $q_2 \dots q_m(q_1 q_2 \dots q_m)^{-1} = q_1^{-1} \in P_i$ for some $i \in I$.
And so we get a chain of adjacent chambers:
$B \sim Bq_m \sim Bq_{m-1}q_m \sim \dots \sim Bq_2 \dots q_m \sim Bq_1 q_2 \dots q_m = Bx$.
Hence the chamber graph is connected.

Now let us suppose that $\mathcal{C}$ is connected and suppose for contradiction that:

$$H = \langle P_i \mid i \in I \rangle \neq G.$$

This means we can choose $x \in H$ and $y \in G$ such that $y \notin H$.
$\mathcal{C}$ is connected and so we can find a minimal path from Bx to By, represented by $(t_1, t_1, \ldots, t_n)$ with $t_0 = x$, $t_n = y$ and $Bt_i \sim Bt_{i+1}$.
As $x \in H$ and $y \notin H$ there must be some i, with $1 \leq i \leq n$, such that $Bt_i \sim Bt_{i+1}$, $t_i \in H$ and $t_{i+1} \notin H$.
$Bt_i \sim Bt_{i+1}$ and so $t_{i+1}t_i^{-1} \in P_j \leq H$.
Hence we have t_i and $t_{i+1}t_i^{-1} \in H$, therefore $t_{i+1} = t_{i+1}t_i^{-1}t_i \in H$.
This gives us a contradiction as we had assumed $t_{i+1} \notin H$.
Thereupon we must have that $H = G$. □

5.1 In the Case of Buildings

The unipotent radical subgroup, U, of a linear algebraic group, H, is the subgroup formed by all unipotent elements in the radical of H. Where the radical is the connected component of the maximal normal solvable subgroup, and unipotent elements are those for which $(x-1)^n = 0$ for some $n \in \mathbb{N}$.
Here we define U to be the unipotent radical of B. It suffices in all cases in this project to take $U = O_2(B)$. Where O_p of a group is defined to be the largest normal p-subgroup, for p a prime. p-subgroups are those for which every element has order that is a power of p.
In particular, in the cases of $GL_3(2)$ and $GL_4(2)$, B is actually a self-normalizing 2-group, and so $B = U$.

Remark 5.14. *Suppose the B in Remark 5.10 is that of a (B, N)-pair, and so the chamber graph we are forming is the chamber graph of a building. In [24], Ronan proves that U acts sharply transitively on the set of chambers opposite γ_0. That is, for any two chambers X and Y in the last disc of $\mathcal{C}$ there exists a unique $u \in U$ such that $X^u = Y$.*
As discussed above, all the examples in Section 6 have the property that $B = U$. We already knew that each disc is a union of B-orbits by Theorem 5.12, and now know that B acts transitively on the last disc. Consequently, in the examples we see the last disc will be a single B-orbit, further because the action is sharply transitive we know that the size of the last disc equals size as B.

Recall that by Remark 5.11, we identify B-orbits with double cosets. In the case of buildings we have a Weyl group W, and so by the Bruhat decomposition the set of double cosets can be given by $\{BwB \mid w \in W\}$, with $BwB \neq Bw'B$ for $w \neq w'$. Hence the set of double cosets is in one to one correspondence with the elements of the Weyl group.

Remark 5.15. *When forming the chamber graph of the Weyl group, W, we take chambers to be the elements of W. By the Bruhat decomposition each element of the Weyl group corresponds to a single B-orbit of our group G. As we shall see the chamber graph of W has the same underlying structure as the chamber graph of G. That is the graphs will have the same shape but in place of elements of the Weyl group we have B-orbits. We see an example of this in Section 6.2.*

We have claimed that the chamber graph gives us another way to view the building, and so we would hope that the chamber graph encodes the same information as the building. As we have seen in Section 3, the Weyl group of a building is generated by the type preserving automorphisms of an apartment. So to recover the Weyl group from the chamber graph we first need a way to find an apartment.

Remark 5.16. *We can find apartments of the chamber graph by taking the geodesic closure of two chambers at maximal distance. From this, we can recover the Weyl group, W, by taking all automorphisms of the apartment which preserve type.*
This means that in moving between the chamber graph and the building we lose no information on apartments and Weyl groups.

We are particularly interested in apartments partly because they are a special subgraph of our chamber graphs. We can learn a lot about the chamber graph even if we only have the graph of the apartment.

Lemma 5.17. *For two chambers in an apartment, the distance between them in the apartment itself is equal to the distance between them in the whole chamber graph.*

Proof. This property follows from one of the ways we can find apartments; by taking the geodesic closure of two opposite chambers.
Find the apartment by picking up all chambers lying on any geodesic between A and B.
Suppose x and y are chambers lying in this apartment. Then there exists a geodesic length, say n, in the chamber graph given by $A = x_0, x_1, \ldots, x_n = B$ with $x_i = x$ and $x_j = y$ for some i, j, $1 < i, j < n$.
Suppose without loss that $i < j$. Then the distance between x and y in the apartment is $j - i$.
Suppose for contradiction that the distance between x and y is shorter in the chamber graph with some path $x = y_0, \ldots, y_k = y$ with $k < (j - i)$.
Then $x_0, \ldots, x_i, y_1, \ldots, y_{k-1}, x_j, x_{j+1}, \ldots, x_n$ is a path in the chamber graph of shorter distance between A and B than in the geodesic. Thus by the definition of geodesic, we reach a contraction. □

6 Examples of Small Chamber Graphs

In Sections 6.1 and 6.2, we use Example 5.8 and Remark 5.10 to form two relatively small Chamber graphs. In both these cases we can use Theorem4.2 to show that the B we find is actually the B of a (B, N)-pair. Hence by Example 4.8 we find that these Chamber graphs are in fact buildings.

We calculate $GL_3(2)$ explicitly by hand using the individual chambers, which we then split into B-orbits. $GL_4(2)$ is calculated using the computer package MAGMA, and its chamber graph is drawn in terms of the B-orbits.
In the first subsection, we demonstrate that we lose no information in terms of apartments and Weyl groups when moving between chamber graphs and buildings. In the second section, we get to see the role the Weyl group plays.

6.1 $GL_3(2)$

Here we view the chamber graph of $GL_3(2)$ in two ways. Firstly in terms of flags, an example of a simplicial complex to which we associate a geometry. Secondly by taking a more group-theoretic approach by looking at a minimal parabolic system.

Consider $G = GL_3(2)$ viewed as the vector space of 3×3 invertible matrices over $\mathbb{Z}_2$. We form a geometry by taking Γ to be the set of all proper non-zero subspaces of V with the type of a space given by its dimension and so $I = \{1, 2\}$. Hence the rank of this system is 2. We define the incidence relation $*$ for $a \neq b$ by $a * b \iff \{a \subset b \text{ or } b \subset a\}$, resultantly this satisfies the condition that $a * b$ implies $\tau(a) \neq \tau(b)$. With this set up a flag is a set of subspaces of G totally ordered by inclusion. And so a maximal flag, or chamber, is a 1 dimensional space incident with a 2 dimensional space.
We shall call the 1-dimensional subspaces points, and the 2-dimensional subspaces lines. Consequently a point is of the form $\langle a \rangle$, a line $\langle a, b \rangle$ for $a \neq b$. The maximal flags are $\langle a \rangle < \langle a, b \rangle$, for $a, b \in \mathbb{Z}_2^3 \backslash \{\underline{0}\}$. Two maximal flags, X and Y, are i adjacent if $\tau(X \cap Y) = T \backslash \{i\}$. We have 7 points and 7 lines given below:

Points $= \{\ \langle(001)\rangle,\ \langle(010)\rangle,\ \langle(011)\rangle,\ \langle(100)\rangle,\ \langle(101)\rangle,\ \langle(110)\rangle,\ \langle(111)\rangle\ \}$

$$\text{Lines} = \left\{ \begin{array}{l} L_1 = \langle(001),(010)\rangle,\ L_2 = \langle(001),(100)\rangle,\ L_3 = \langle(001),(110)\rangle,\ L_4 = \langle(010),(100)\rangle, \\ L_5 = \langle(010),(101)\rangle,\ L_6 = \langle(011),(100)\rangle,\ L_7 = \langle(011),(110)\rangle \end{array} \right\}$$

We can represent the incident relation in the Fano plane below. In the graph a point lies on a line if and only if it is incident with the line.

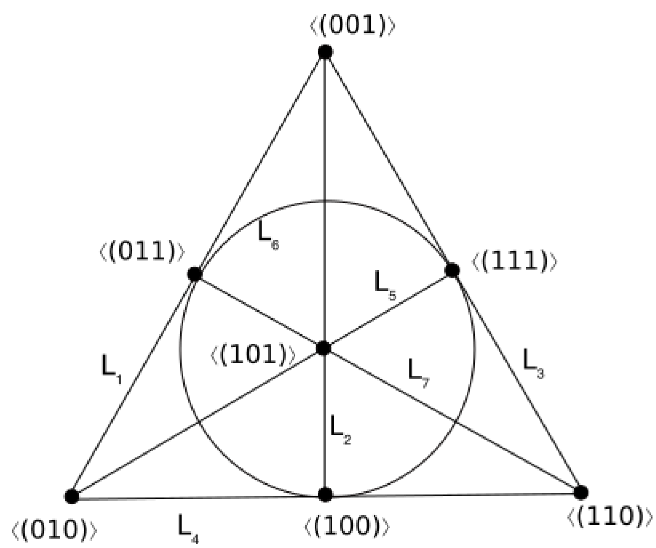


Figure 5: Incidence graph

Remark 6.1. *We have already seen the Fano plane before in Section 3.2. There we showed this is a building with Weyl group containing $Sym(3)$ as a subgroup. We now know from Theorem 4.2 that $Sym(3)$ is indeed the Weyl group of this building. Forming the chamber*

graph will give us another way to view this building.
We verify remark 5.16, in particular we show that we can recover the chambers of the apartment Σ we found in Section 3.2.

From the Fano plane we find that the set of all the flags is $\{\gamma_0, \dots \gamma_{20}\}$ with the γ_i as below.

$\gamma_0 = \langle(001)\rangle < \langle(001),(010)\rangle$
$\gamma_1 = \langle(001)\rangle < \langle(001),(100)\rangle$
$\gamma_2 = \langle(001)\rangle < \langle(001),(110)\rangle$
$\gamma_3 = \langle(010)\rangle < \langle(001),(010)\rangle$
$\gamma_4 = \langle(011)\rangle < \langle(001),(010)\rangle$
$\gamma_5 = \langle(100)\rangle < \langle(001),(100)\rangle$
$\gamma_6 = \langle(101)\rangle < \langle(001),(100)\rangle$
$\gamma_7 = \langle(110)\rangle < \langle(001),(110)\rangle$
$\gamma_8 = \langle(111)\rangle < \langle(001),(110)\rangle$
$\gamma_9 = \langle(010)\rangle < \langle(010),(100)\rangle$
$\gamma_{10} = \langle(010)\rangle < \langle(010),(101)\rangle$
$\gamma_{11} = \langle(011)\rangle < \langle(011),(100)\rangle$
$\gamma_{12} = \langle(011)\rangle < \langle(011),(101)\rangle$
$\gamma_{13} = \langle(100)\rangle < \langle(010),(100)\rangle$
$\gamma_{14} = \langle(100)\rangle < \langle(011),(100)\rangle$
$\gamma_{15} = \langle(101)\rangle < \langle(010),(101)\rangle$
$\gamma_{16} = \langle(101)\rangle < \langle(011),(101)\rangle$
$\gamma_{17} = \langle(110)\rangle < \langle(010),(100)\rangle$
$\gamma_{18} = \langle(110)\rangle < \langle(011),(101)\rangle$
$\gamma_{19} = \langle(111)\rangle < \langle(010),(101)\rangle$
$\gamma_{20} = \langle(111)\rangle < \langle(011),(100)\rangle$

γ_j and γ_k are i adjacent if they have the same subspaces for all dimensions other than i.
$\tau(\text{point}) = 1$ and $\tau(\text{line}) = 2$. And so Γ_1 is the set of all points, Γ_2 is the set of all lines. The rank of the system is $|I| = 2$.

$$\begin{aligned} \gamma_j, \gamma_k \text{ are 1 adjacent } &\iff \tau(\gamma_j \cap \gamma_k) = I \backslash \{1\} = 2 \\ &\iff \gamma_j \cap \gamma_k \text{ share the same line} \end{aligned}$$

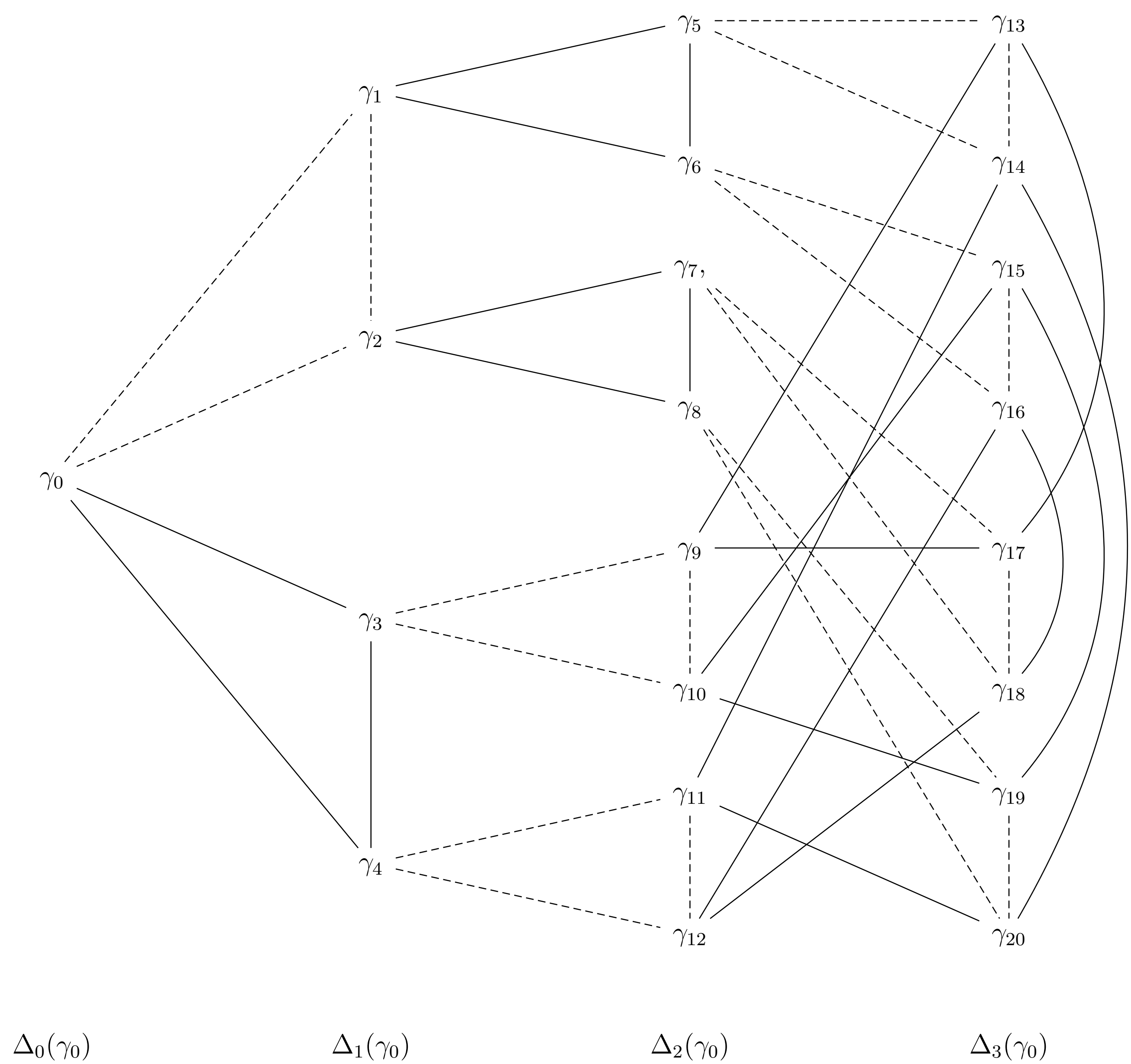
Similarly, γ_j, γ_k are 2 adjacent $\iff$ they share the same point.

$$\begin{aligned} \gamma_0 = \langle(001)\rangle < \langle(001),(010)\rangle &\sim_1 \langle(010)\rangle < \langle(001),(010)\rangle = \gamma_3 \\ &\sim_1 \langle(011)\rangle < \langle(001),(010)\rangle = \gamma_4 \end{aligned}$$

Hence γ_0, γ_3 and γ_4 are all type 1 neighbours.

$$\begin{aligned} \gamma_0 = \langle(001)\rangle < \langle(001),(010)\rangle &\sim_2 \langle(001)\rangle < \langle(001),(100)\rangle = \gamma_1 \\ &\sim_2 \langle(001)\rangle < \langle(001),(110)\rangle = \gamma_2 \end{aligned}$$

Hence γ_0, γ_1 and γ_2 are all type 2 neighbours.
Therefore $\Delta_1(\gamma_0) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$.
Continuing in a similar way we find the entire chamber graph of $GL_3(2)$.
The chamber graph is given below showing the connection between flags.
Normally the lines would be labelled with i if the adjacency was of type i. In order to prevent a labelling nightmare in the last disc we instead use a solid edge for type 1-adjacency and a dashed edge for type 2.



Form the chamber graph we see the diameter is three.

We already know from Theorem 4.2 that the B of the (B, N)-pair of $GL_3(2)$ is the set of uni-upper triangular matrices. Can we verify this using Example 4.9 to find $B = Stab_G(\gamma_0)$? Let $A \in B,\ A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} g & h & i \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \Rightarrow g = h = 0$ and $i = 1$.

$\begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} d & e & (f{+}1) \end{pmatrix} \in \langle (001), (010) \rangle = \{(001), (010), (011)\} \Rightarrow d = 0.$

$A \in GL_3(2)$ and therefore must have non zero determinant, so $\det(A) = 1$ as we're working over $\mathbb{Z}_2$. Expanding along the bottom row and using the fact that $g = h = 0$ and $i = 1$ we find that $\det(A) = ae - bd = 1$. Also $d = 0$ gives us that $ae = 1$ and so $a = e = 1$.
Consequently B is indeed the subgroup of G given by uni-upper triangular matrices.

$$B = \left\{ \begin{array}{llll} B_1 = \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, & B_2 = \begin{pmatrix} 1&0&1\\0&1&0\\0&0&1 \end{pmatrix}, & B_3 = \begin{pmatrix} 1&1&0\\0&1&0\\0&0&1 \end{pmatrix}, & B_4 = \begin{pmatrix} 1&1&1\\0&1&0\\0&0&1 \end{pmatrix}, \\ B_5 = \begin{pmatrix} 1&0&0\\0&1&1\\0&0&1 \end{pmatrix}, & B_6 = \begin{pmatrix} 1&0&1\\0&1&1\\0&0&1 \end{pmatrix}, & B_7 = \begin{pmatrix} 1&1&0\\0&1&1\\0&0&1 \end{pmatrix}, & B_8 = \begin{pmatrix} 1&1&1\\0&1&1\\0&0&1 \end{pmatrix} \end{array} \right\}$$

We know by Remark 5.14 that the last disc should be a single B-orbit. Can we show this? We need to show that given an element of the last disc, say γ_{13}, we can map it to all other elements of $\Delta_3(\gamma_0)$ using the elements of B.
First consider γ_{13} and B_2:
$\begin{pmatrix} 1&0&0 \end{pmatrix} B_2 = \begin{pmatrix} 1&0&1 \end{pmatrix} \Rightarrow \langle(100)\rangle B_2 = \langle(101)\rangle$
$\begin{pmatrix} 0&1&0 \end{pmatrix} B_2 = \begin{pmatrix} 0&1&0 \end{pmatrix}$ and $\begin{pmatrix} 1&1&0 \end{pmatrix} B_2 = \begin{pmatrix} 1&1&1 \end{pmatrix} \Rightarrow \langle(010),(100)\rangle B_2 = \langle(101),(010)\rangle$
And so γ_{13} is mapped to γ_{15} by B_2.
Similarly γ_{13} is mapped to: γ_{13} by B_1, γ_{17} by B_3, γ_{19} by B_4, γ_{14} by B_5, γ_{16} by B_6, γ_{18} by B_7 and γ_{20} by B_8. And so the last disc, $\Delta_3(\gamma_0)$, is a B-orbit.

Using the same method as above we find $\Delta_2(\gamma_0)$ is the union of two B-orbits, $\{\gamma_5, \gamma_6, \gamma_7, \gamma_8\}$ and $\{\gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}\}$. $\Delta_1(\gamma_0)$ is also the union of 2 B-orbits, $\{\gamma_1, \gamma_2\}$ and $\{\gamma_3, \gamma_4\}$.

Can we verify Remark 5.16 and recover the apartment and Weyl group from the chamber graph? Comparing the two diagrams of the Fano plane in this Section and in Section 3.2, we see that the apartment we called Σ corresponds to,

$$\{\emptyset, \langle(010)\rangle, \langle(001)\rangle, \langle(110)\rangle, L_1, L_3, L_4, \gamma_0, \gamma_2, \gamma_9, \gamma_3, \gamma_7, \gamma_{17}\}.$$

The only one of these chambers that lies in the last disc is γ_{17}, so we take the geodesic closure $\overline{\{\gamma_0, \gamma_{17}\}}$.
γ_{17} has two neighbours in the second disc, namely γ_7 and γ_9. γ_7's only neighbour in the first disc is γ_2, and γ_9's is γ_3. Finally the only neighbour of γ_3 and γ_2 in the zero disc is γ_0. Thus $\overline{\{\gamma_0, \gamma_{17}\}} = \{\gamma_0, \gamma_2, \gamma_9, \gamma_3, \gamma_7, \gamma_{17}\}$.
We now take all type 1 and type 2 objects required to turn this into a subcomplex, that is all objects that make up the chambers $\gamma_0, \gamma_2, \gamma_9, \gamma_3, \gamma_7$ and γ_{17}. Doing this gives the set $\{\emptyset, \langle(010)\rangle, \langle(001)\rangle, \langle(110)\rangle, L_1, L_3, L_4, \gamma_0, \gamma_2, \gamma_9, \gamma_3, \gamma_7, \gamma_{17}\}$. Hence we have been able to recover the same apartment from the chamber graph as we found in the building.
We could now recover the Weyl group from this apartment as discussed in Remark 5.16.

We have seen the chamber graph from a flag point of view. Let us now try emulating Example 5.8 making use of Remark 5.10 to identity the chambers with the cosets of B.
Consider the chamber $\gamma_0 = \langle(001)\rangle < \langle(001),(010)\rangle$. We have already calculated $B = Stab_G(\gamma_0)$, and in a similar way, we can calculate $P_1 = Stab_G(\langle(001)\rangle)$ and $P_2 = Stab_G(\langle(001),(010)\rangle)$.
We find that

$$P_1 = \left\{ A = \begin{pmatrix} a&b&c\\d&e&f\\0&0&1 \end{pmatrix} \in \mathbb{Z}_2^{3\times 3} \;\middle|\; det(A) \neq 0 \right\},$$

$$P_2 = \left\{ A = \left(\begin{smallmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{smallmatrix}\right) \in \mathbb{Z}_2^{3\times 3} \;\middle|\; det(A) \neq 0 \right\}.$$

By considering the number of possible linearly independent rows/columns, we find that $|G| = (2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 168$, for each i $|P_i| = (2^3 - 2)(2^3 - 2^2) = 24$, and $|B| = 8$.

$|G| = 168 = 2^3 \cdot 3 \cdot 7$, and so B is a Sylow 2 subgroup of G. We know that B is the B of a (B, N)-pair by Example 4.2 and so as discussed in Theorem 4.11 B is self normalising, that is $B = N_G(B)$. Hence B the normalizer of a Sylow 2 subgroup.
In addition by Example 4.9 we know $B = Stab_G(\gamma_0)$.

For each i, $[P_i, B] = 3$ and so by 5.4 B is itself a maximal subgroup of P_i and can be contained in no other maximal subgroup, and so P_i is a minimal parabolic subgroup.

$G = \langle P_1, P_2 \rangle$ with $P_1, P_2 \lneq G$, consequently $\{P_1, P_2\}$ is a minimal parabolic system.

As a result this satisfies the conditions in Example 5.8.
By Lagrange's theorem we know that the number of cosets is $\frac{|G|}{|B|} = \frac{168}{8} = 21$, which agrees with our previous calculation of the number of chambers. Recall we say two cosets Bg and Bh to be i-adjacent exactly when $gh^{-1} \in P_i$. If we complete the chamber graph with this method then we will get the same picture but with the point γ_i being replaced by the cosets of B.

$[P_i : B] = 3$, so the transversal of P_i and B will contain three elements, one being the identity. The other two will be those that map B to its two neighbours of type i. Therefore each chamber has two neighbours of each type, again agreeing with our calculations above in terms of chambers.

6.2 $GL_4(2)$

The number of chambers of $GL_4(2)$ is much greater than that of $GL_3(2)$, and so instead of computing its chamber graph by hand we use the computer program MAGMA. The code for this was adapted from [3].
We form the chamber graph of the Weyl group by hand and use this to demonstrate Remark 5.15. In addition, we use it to calculate the size of the discs for $GL_4(2)$ without a computer.

6.2.1 Notation

As we saw in Remark 5.10, the right cosets of B are in one to one correspondence with the chambers of Γ and so we identify these.
And by remark 5.11 the B-orbits are identified with double cosets.
The code uses the command $DB := DoubleCosetRepresentatives(G, B, B)$ to find a sequence of all double coset representatives. A B-orbit represented by $BDB[i]$ is then stored as i, where $i = 1$ corresponds to B. We take B to be the fixed chamber. The code uses this method of recording B-orbits to save on storage space. This is required because although the chamber graphs we are looking at are relatively small the code was first developed and

used for much larger groups.
The code stores $NeighboursofB$ as an ordered sequence, ordered by type of adjacency, these show the chambers that are adjacent to B. $Neighbours$ is a sequence of sequences which stores the neighbours for each chamber. The i^{th} entry, $Neighbours[i]$, is a sequence of double coset representatives. If j is an element in $Neighbours[i]$ then the two B-orbits represented by $BDB[i]$ and $BDB[j]$ are adjacent. Once again this sequence is ordered by type of adjacency. With the type 1 neighbours appearing at the start, and then all of those of type 2 and so on. Hence the position of j in the sequence $Neighbours[i]$ indicates which type of neighbours $BDB[i]$ and $BDB[j]$ are.
For a more detailed description of the method see the appendix for an annotated version of the code.

6.2.2 Calculation

We form the system in Example 5.8 using a similar approach to that in Section 6.1. Again we make use of Remark 5.10 to identify the chambers with cosets of B.
There are now 3 types of objects. The dimension 1,2 and 3 subspaces are the type 1,2 and respectively 3 type objects. The incidence relation is given by subspace inclusion, and so a chamber is a 1 dimensional subspace lying in a 2 dimensional subspace in turn lying in a 3 dimensional subspace. Sometimes denoted as {point < line < plane}.
Recall that two chambers are i adjacent if they differ only in the type i space. And so we take an arbitrary {point < line < plane} in the graph and calculate the stabilizer of space. We then take pairwise intersections of these stabilizers to find our P_is.

Consider $\gamma_0 = \langle(0001)\rangle < \langle(0001),(0010)\rangle < \langle(0001),(0010),(0100)\rangle = U_1 < U_2 < U_3$. $P_{23} = Stab_G(U_1)$, $P_{13} = Stab_G(U_2)$ and $P_{12} = Stab_G(U_3)$. We find that:
$P_{23} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $P_{13} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$, $P_{12} = \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$.
Take B to be the subgroup given by all matrices of $GL_4(2)$ which fix the entire system $U_1 < U_2 < U_3$, and so $B = \begin{pmatrix} 1 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Using the fact that the elements of B must be invertible, $B = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}$.
The elements labelled by stars are arbitrary provided the resulting matrices are non singular.

Let us set $P_1 = P_{12} \cap P_{13}$, $P_2 = P_{12} \cap P_{23}$ and $P_3 = P_{13} \cap P_{23}$.
By considering the number of rows that can be linearly independent, we find that
$|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$, $|B| = 2^6$ and $|P_i| = 2^6 \cdot 3$.
Therefore, B is a Sylow 2 subgroup of G. B is also the B of a (B, N) pair by Example 4.2 and so B is self normalizing by Theorem 4.11. Thus B is the normaliser of a Sylow 2 subgroup of G.
Again using Example 5.4 we find that B is itself a maximal subgroup in each P_i, and can be contained in no other maximal subgroups. And so each P_i is a minimal parabolic subgroup with respect to B.
$G = \langle P_1, P_2, P_3\rangle$, and G cannot be generated by any proper subset $\{P_1, P_2, P_3\}$.

Therefore we satisfy all the conditions in Example 5.8. Accordingly we can identify the chambers with the cosets of B by Remark 5.10.

We can work out the number of chambers working about the number of cosets, and so the number of chambers is $\frac{|G|}{|B|} = \frac{20160}{64} = 315$.
$[P_i : B] = 3$ for each i, with three representative elements in the transversal, $\{id, u, v\}$. And so each chamber, Bx, has two neighbours of type i given by Bux and Bvx.

Applying the computer code for $GL_4(2)$ we find the diameter of the chamber graph is six, and it has the disc structure is shown below.

i^{th} Disc	1	2	3	4	5	6
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	48	80	96	64
Number of B-orbits	3	5	6	5	3	1

As we can see the last disc is a single B-orbit which agrees with Remark 5.14.

The collapsed adjacency graph of the Chamber graph of $GL_4(2)$ is shown below, each B-orbit is given a representative label as discussed in the section on notation. Two B-orbits X and Y are joined by an arrow labelled with i from X to Y if and only if each chamber in the B-orbit X has i neighbours in the B-orbit Y.

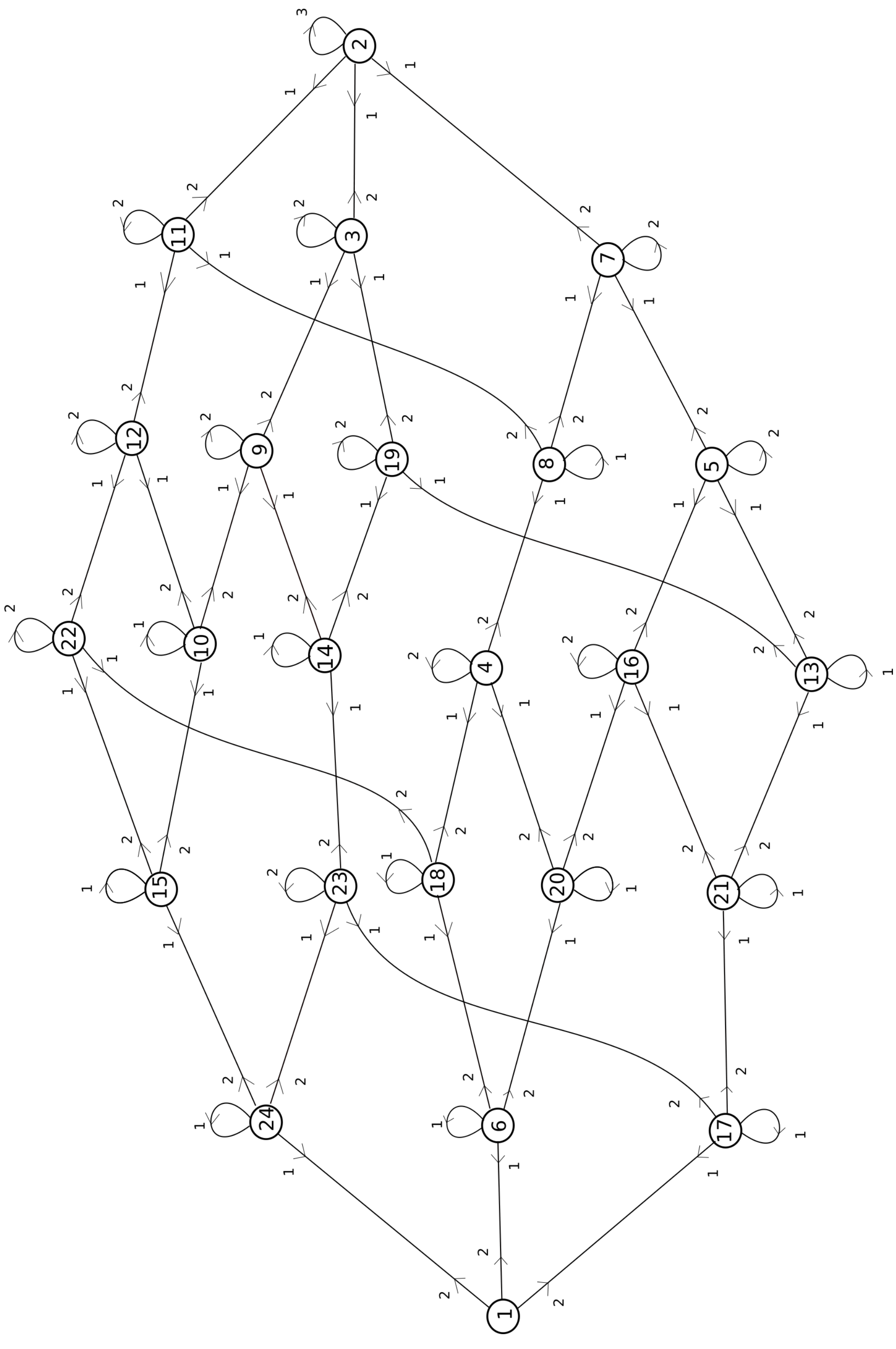


Figure 6: Chamber Graph of $GL_4(2)$

The type of adjacency of each neighbour in not included in this diagram in order to avoid confusion between the type of neighbours and the number of neighbours. However this information is saved in MAGMA.

As we have already discussed, we know the Weyl group controls a lot of the structure of the chamber graph, and we see an example of this here.
We could have used the Weyl group to calculate the size of the discs in the chamber graph completely without computer. First we need to calculate the chamber graph of the Weyl group.
Using Theorem 4.2 we have the for $G \cong GL_4(2)$ the (B, N)-pair is given by:

- B = Upper triangular 3×3 matrices
- N = matrices with one non zero entry in each row/column
- $B \cap N$ = Diagonal matrices

The Weyl group of G is given by $N/B \cap N$, and so $W \cong Sym_4$, as discussed in Remark 4.4. Calculating the chamber graph of this gives the structure of an apartment within the chamber graph.

We know W has generating set given by $S = \{(1,2),(2,3),(3,4)\} = \{s_1, s_2, s_3\}$, by the running Example 2.4.
As discussed by Ronan in [23], we can form the chamber system of W by taking the chambers to be the elements of W. Set $w_1, w_2 \in W$ to be i adjacent if $w_1 = w_2 s_i$.
As a result, for a chamber g, the type 1 neighbour is $g(1,2)$, the type 2 is $g(2,3)$ and the type 3 is $g(3,4)$. We take γ_0 to be the identity.
The chamber graph of $Sym(4)$ is shown below. Two vertices, which represent chambers, are joined by an edge labelled by i if and only if they are i-adjacent.

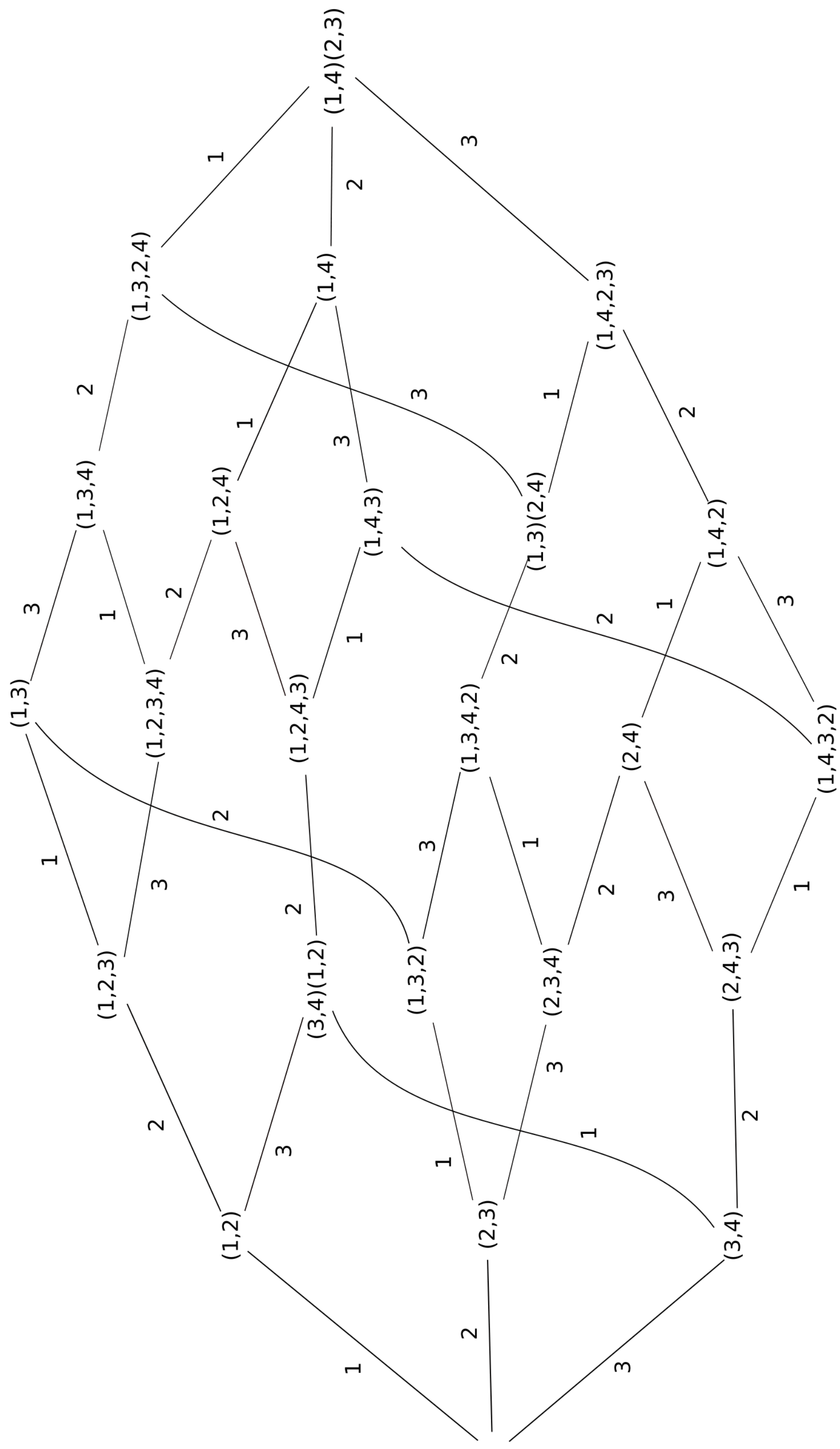


Figure 7: Chamber Graph of $Sym(4)$

As discussed in Remark 5.15, we know that the chamber graph of the Weyl group should have the same structural shape as the chamber graph of the whole group. We can see this holds true for the chamber graphs for $GL_4(2)$ and $Sym(4)$. In particular, each element of the Weyl group corresponds to a B-orbit in the chamber graph of $GL_4(2)$.

We claimed that, from the chamber graph of the Weyl group, we could calculate the sizes of the discs of $GL_4(2)$ by hand. In order to do this, we need to know the lengths of the elements in each disc of the Weyl group.

Remark 6.2. *We are taking $\gamma_0 = id$, and $w \sim_i v \iff w = vs_i$.*
Hence the first disc is given by, $\{s_i \mid s_i \in S\}$, that is all elements in $Sym(4)$ of length one.
Again, by the joining relation, we can see that the second disc is given by all elements in the set $\{s_i s_j \mid s_i, s_j \in S\}$, that we have not already seen in disc 0 or disc 1. Therefore disc 2 is given by all elements of length two.
Similarly we see that disc i is given by the elements of length i.

This means the diameter of the apartment is the length of the longest element in $Sym(4)$. As we know from Section 2 this is $(1,4)(2,3)$, and is unique.

We can now apply the following formula from [26]:

$$|\Delta_i(\gamma_0)| = q^{\ell(w)} \times |\Delta'_i(1)|.$$

Here $\Delta'_i(1)$ denotes the i^{th} disc of the Weyl group, q is the characteristic of the field, and $\ell(w)$ is the length of an element in the i^{th} disc of the chamber graph of the Weyl group.
Using the chamber graph on the previous page, we can see that $Sym(4)$ has the following disc structure.

i^{th} Disc	1	2	3	4	5	6
$\lvert\Delta'_i(\gamma_0)\rvert$	3	5	6	5	3	1

We already know the length of a word in each disc is equal to the disc's number by Remark 6.2. The field we are working over is $\mathbb{Z}_2$. Let us apply the formula and check we get the same disc sizes as from Magma.

i^{th} Disc	1	2	3	4	5	6
$\lvert\Delta'_i(\gamma_0)\rvert$	3	5	6	5	3	1
$2^i \cdot \lvert\Delta'_i(1)\rvert$	$2^1 \cdot 3 = 6$	$2^2 \cdot 5 = 20$	$2^3 \cdot 6 = 48$	$2^4 \cdot 5 = 80$	$2^5 \cdot 3 = 96$	$2^6 \cdot 1 = 64$
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	48	80	96	64

We can see that the numbers in the last two rows agree with one another!

7 Mathieu Groups

The Mathieu groups were first discovered by Émile Léonard Mathieu between 1860 and 1873. Miller disproved the existence of these groups in 1898 but then only two years later retracted this and went on to prove that these groups not only exist but are simple. Throughout this project, we trust Miller's second instinct.

Definition 7.1. *A group G is called k transitive if for any two sets $\{x_1, \ldots, x_k\}$ and $\{y_1, \ldots, y_k\}$ with $x_i \neq x_j$ and $y_i \neq y_j$ for $i \neq j$, we have that there is some $g \in G$ such that g maps x_i to y_i for each i, $1 \leq i \leq k$.*
We call G sharply k transitive if such a g is unique.

Clearly if group is k transitive then it is l transitive for $l \leq k$.
The Mathieu groups were first discovered because of a search for multiply transitive groups (groups that are k transitive for some $k \in \mathbb{N}$).
M_{24} is 5 transitive, M_{23} 4 transitive, and M_{22} 3 transitive. M_{12} is sharply 5 transitive, and M_{11} is sharply 4 transitive.
Being multiply transitive is a rather strong property, in fact the only 4-transitive groups are $Sym(n)$ for $n \geq 4$, $Alt(n)$ for $n \geq 6$, M_{24}, M_{23}, M_{12}, and M_{11}. This is not easy to prove and the proof, in [2], relies on the classification of finite simple groups.

The classification of finite simple groups categorises all of the finite simple groups into 18 infinite but countable families, apart from 26 so called sporadic groups which fail to fit into a logical system. The Mathieu groups are all in this set of sporadic simple groups.
The descriptor "sporadic" is partially due to Burnside who was the first to use this term in a group theoretic sense when he was describing the Mathieu groups.

Mathieu groups also have relevance in coding theory. The Extended Binary Golay Code G_{24} which was used by NASA in Voyager 1 and 2 is closely linked to M_{24}. G_{24} uses a word of bit length 24 to encode 12 bits of information. Words of weight eight in this code are called octads of which there are 759. M_{24} is an automorphism group of this code and is transitive on the set of octads.

Perhaps the most helpful way to view the Mathieu groups is as automorphism groups of Steiner systems, which we now define.

Definition 7.2. *An $S(a, b, c)$ Steiner System a set Ω of size c, and a collection of subsets of size b called blocks with the property that any a distinct elements of Ω lie in a unique b-element subset.*

Remark 7.3. *We can think of the Fano plane as drawn in Section 6.1 as an example of a Steiner system. Recall that the points were labelled by 3-tuples with entries coming from $\mathbb{Z}_2$. Looking back at the diagram we can notice that a line is made up of three points, but uniquely determined by just two. This is because the third point is the sum of the other two. For any two points, there is a unique line that they lie on. We can define this set up to be a Steiner system of the form $S(2, 3, 7)$.*
For example, given $\langle(001)\rangle$ and $\langle(010)\rangle$ we know these points lie in the unique line $\langle(001), (010), (011)\rangle$.

The correspondence between each Mathieu group and a certain Steiner system is given below.

Definition 7.4.
$M_{11} = \{\rho \in S_{12} \mid \rho(s) \in S(4, 5, 11) \;\; \forall s \in S(5, 6, 12)\}$
$M_{12} = \{\rho \in S_{12} \mid \rho(s) \in S(5, 6, 12) \;\; \forall s \in S(5, 6, 12)\}$
$M_{22} = \{\rho \in S_{22} \mid \rho(s) \in S(3, 6, 22) \;\; \forall s \in S(3, 6, 22)\}$

$M_{23} = \{\rho \in S_{23} \mid \rho(s) \in S(4,7,23) \;\; \forall s \in S(4,7,23)\}$
$M_{24} = \{\rho \in S_{24} \mid \rho(s) \in S(5,8,24) \;\; \forall s \in S(5,8,24)\}$

For arbitrary $a \leq b \leq c$ there will not necessarily exist a non trivial Steiner system $S(a,b,c)$, but once we have a given system it is relatively easy to build another as shown below.

Theorem 7.5. *If a Steiner System of the form $S(a,b,c)$ exists, then so does one of the form $S(a-1,b-1,c-1)$.*

Proof. Suppose we have an $S(a,b,c)$ Steiner System.
Consider a point $x \in \Omega$ and and discard all blocks that which do not contain x. We call this new collection of blocks the "Edited Steiner System".
Suppose we take any set of a points containing x. There must be a unique block in the "Edited Steiner System" containing these a points.
If we now disregard the point x we are left with the set $\Omega\backslash\{x\}$ which contains $c-1$ elements. The "Edited Steiner System" contains blocks with $b-1$ points and we have the property that any $a-1$ points of $\Omega\backslash\{x\}$ lie in a unique block.
Consequently we have formed an $S(a-1,b-1,c-1)$ Steiner System. □

Using Theorem 7.5 it suffices to show that $S(5,8,24)$ and $S(5,6,12)$ exists in order to show that all the Steiner Systems above exist. For the existence of $S(5,8,24)$ see [9], and for the existence of $S(5,6,12)$ see [11]

We will look at each of the Mathieu groups in more depth in later sections.
However, we look no further into M_{11}. Here we are using the geometries in [24] to form chamber graphs. We see in Section 11 the way Ronan and Stroth define these geometries. Requiring that $O_p(P_i) \neq 1$ means in the case of $p = 2$ there is only one such P_i, and so we cannot generate M_{11} by such P_i. Similar reasons for other primes mean we are not able to find a system of M_{11} satisfying all the conditions set by Ronan and Stroth.
There are, however, many other geometries of M_{11}, and some of their chamber graphs are calculated in [26].

8 Chamber Graphs of Sporadic Groups

We want to emulate the calculations of $GL_3(2)$ and $GL_4(2)$ when our groups are not of Lie type, for example the sporadic Mathieu groups. We do this by altering the geometric requirements for a minimal parabolic system.
We form the systems in [24] using the following construction.

Let G be one of the Sporadic groups and p a prime dividing the order of G. We take B to be the normalizer of a Sylow-p subgroup of G. We find a set of subgroups $\{P_1, \ldots, P_n\}$, $I = \{1, \ldots, n\}$ such that

(i) $O_p(P_i)$ is not trivial for each $i \in I$. (Where $O_p(P_i)$ is the largest normal p-subgroup of P_i),

(ii) B lies in exactly one maximal subgroup of P_i for each $i \in I$,

(iii) $\{P_1, \ldots, P_n\}$ is a minimal set of subgroups which generate G.

In particular we are interested in geometric systems. That is sets of P_i's such that $P_J \cap P_K = P_{J\cap K}$, where $P_J = \langle P_j \mid j \in J \subseteq I\rangle$ and $P_\emptyset = B$.
This condition is added so that the complex we get is in fact a simplicial complex.

As in Example 5.8 we take the chamber system given by taking chambers to be given by the conjugates of B, and we say that $B^g \sim_i B^g \iff gh^{-1} \in P_i$.
Again making use of Remark 5.10 to identify conjugates of B with cosets.

Note that this will not give us a building in the case of the Mathieu groups as the B we find is not the B of a (B, N)-pair, indeed no (B, N)-pair exists. However as in Example 4.9, B will be the stabilizer of a chamber.
It is interesting to explore how much these chamber graphs will look like those of buildings. We saw in Section 6 that the Weyl group controls a lot of the behaviour of the chamber graph. As sporadic groups do not have a Weyl group we would expect the behaviour of their chamber graphs to be a bit less predictable.
By Remark 5.14, in the case of the chamber graphs of buildings with $B = U$, we have that the last disc is a single B-orbit. For that reason when we calculate the chamber graphs of the geometries of the sporadic groups it seems interesting to see how many B-orbits are in the final disc.
When we have particularly few B-orbits in the last disc we can easily calculate the geodesic closures for each B-orbit. This is motivated by the fact that in the case of buildings this would give us an apartment.

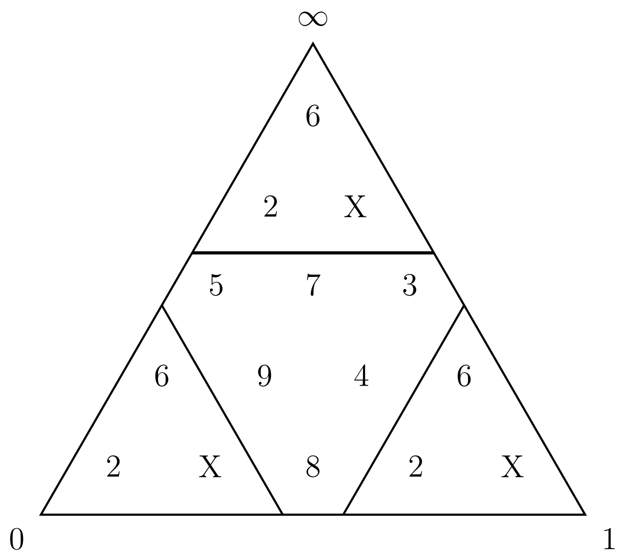
The number of B-orbits in each disc of chamber graphs of these structures has been calculated by Rowley in [26] using CAYLEY. With the use of MAGMA we replicate this data and, in addition, find the number of chambers in each disc.

9 The Kitten

The Kitten was first developed by Conway and Curtis. It gives us a way to quickly find all hexads of the $S(5, 6, 12)$ Steiner system. Part of the advantage of this structure is that it allows us to recognise certain symmetries of hexads. The Kitten mirrors the earlier construction of The MOG which offers a combinatorial way to view the octads of $S(5, 8, 24)$.
The choice of labelling of the Kitten, squares and crosses in this section comes from [29].
We take $\Omega = \{\infty, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X\}$.

The choice of labelling of the points in Ω comes from a way to construct the hexads of $S(5, 6, 12)$ by Conway and Curtis in [11]. We take $P^1(\mathbb{F}_{11})$, the projective line over F_{11}, and consider $Q = \{0, 1, 3, 4, 5, 9\}$ to be the set of points $q \in \mathbb{F}_{11}$ such that $x^2 = q \ mod\ 11$ for some $x \in \mathbb{F}_{11}$ (such points are called quadratic residues). We form the group $L = \langle f, g\rangle$ generated by the two maps on $P^1(\mathbb{F}_{11})$ given by $f(x) = x + 1$ and $g(x) = -\frac{1}{x}$. The set of hexads is given by $S = \{\theta(Q) \mid \theta \in L\}$.

The Kitten is given by the following diagram:



The points ∞, 0 and 1 are called the points at infinity and they have corresponding pictures.

6	X	3
2	7	4
5	9	8

∞ - picture

5	7	3
6	9	4
2	X	8

0 - picture

5	7	3
9	4	6
8	2	X

1 - picture

These grids of points come from taking the triangle in the Kitten that corresponds to the point at infinity along with the central panel.

The diagrams below represent the "lines". A line is any 3 points in a diagram represented by the same symbol (•, ∘ or ×). The first picture shows the horizontal lines, the second the vertical and the third and fourth show the generalized diagonals.

×	×	×
∘	∘	∘
•	•	•

A

×	∘	•
×	∘	•
×	∘	•

B

×	•	∘
∘	×	•
•	∘	×

C

∘	•	×
•	×	∘
×	∘	•

D

We say two lines are parallel if they are two distinct lines that both lie together in one of A, B, C or D. Two lines are said to be perpendicular if one lies in A and the other in B, or one in C and the other in D. We call the union of two perpendicular lines a cross, and the remaining points in a picture a square.

Below is a list of all the squares and crosses, where a square is made of all the points represented by ■ and the crosses by all points represented by ×. We use the notation

$A_\bullet \cup B_\circ$ to mean that the cross is formed by taking the union of the line in A represented by $\bullet$, and the line in B represented by $\circ$.

■	■	×
■	■	×
×	×	×

$A_\bullet \cup B_\bullet$

■	×	■
■	×	■
×	×	×

$A_\bullet \cup B_\circ$

×	■	■
×	■	■
×	×	×

$A_\bullet \cup B_\times$

■	■	×
×	×	×
■	■	×

$A_\circ \cup B_\bullet$

■	×	■
×	×	×
■	×	■

$A_\circ \cup B_\circ$

×	■	■
×	×	×
×	■	■

$A_\circ \cup B_\times$

×	×	×
■	■	×
■	■	×

$A_\times \cup B_\bullet$

×	×	×
■	×	■
■	×	■

$A_\times \cup B_\circ$

×	×	×
×	■	■
×	■	■

$A_\times \cup B_\times$

■	×	■
×	■	×
×	■	×

$C_\bullet \cup D_\bullet$

×	×	■
■	■	×
×	×	■

$C_\bullet \cup D_\circ$

■	×	×
■	×	×
×	■	■

$C_\bullet \cup D_\times$

■	×	×
×	■	■
■	×	×

$C_\circ \cup D_\bullet$

×	■	×
×	■	×
■	×	■

$C_\circ \cup D_\circ$

■	■	×
×	×	■
×	×	■

$C_\circ \cup D_\times$

×	×	■
×	×	■
■	■	×

$C_\times \cup D_\bullet$

×	■	■
■	×	×
■	×	×

$C_\times \cup D_\circ$

×	■	×
■	×	■
×	■	×

$C_\times \cup D_\times$

Remark 9.1. *Note that each of the crosses and squares have some sort of reflective symmetry. Keeping this in mind can speed up the process of finding the hexad containing a given set of points, as we will see below.*

There four ways to forming hexads, given by taking any of the following:

(i) The union of two parallel lines in any of the pictures,

(ii) A point at infinity with a cross in its picture at infinity,

(iii) Two points at infinity and a square in other infinity picture,

(iv) All three points at infinity and a line.

Let us check that we get all hexads from these methods.

1. There are 12 different arrangements of points to give a line, 3 from each of A, B, C and D. The set of lines that come from each (0,1, or ∞) picture are equal. For example, the $A_\bullet$ line in the ∞-picture is the same as the $C_\times$ line in the 0 - picture and the $B_\times$ line in the 1 - picture. Hence the set of points given by the union of 2 parallel lines is the same for any picture. This means that we only need to consider the number of ways we can pick 2 parallel lines, that is the ways to choose 2 lines out of a possible 3 for each of A, B, C and D. And so we get $\binom{3}{2} \times 4 = 12$.

2. We have 3 choices of a point at infinity and for each choice there are 18 choices of crosses, so this method gives us $3 \times 18 = 54$ hexads.

3. We have 3 ways to choose any two points at infinity and 18 choices for a square in the remaining picture at infinity. And so we get $3 \times 18 = 54$ hexads.

4. The only free choice for this method is choosing which line to use. There are 12 lines, and the set of lines for each (0,1, or ∞) picture is the same. And so we get 12 lines from this method.

And so we get $12 + 54 + 54 + 12 = 132$ hexads from these methods. Note no two methods can give the same hexads because they each have different numbers of points at infinity. We know there are $\binom{12}{5}/\binom{6}{5} = 132$ hexads and so we have them all.

Example 9.2. *Let us try finding a unique hexad for a given 5 points. Suppose we have the points* $\{0,\ 1,\ 2,\ 3,\ 4\}$. *This already contains at least 2 points at infinity and could potentially contain all three if the last point is* ∞. *This means that we only have to consider methods (iii) and (iv). We first try considering method (iii). We have two points at infinity namely 0 and 1, and so we need to consider the* ∞*-picture. Let us mark the points we have so far in bold on the* ∞*-picture. We get the following.*

6	X	**3**
2	7	**4**
5	9	8

To complete the hexad we need to add a point to the symbols in bold in such a way that we get a square. Searching among the squares we find the only option is to add in the point 6 (from the square in $A_\bullet \cup B_\circ$*). This gives us the hexad* $\{0,\ 1,\ 2,\ 3,\ 4,\ 6\}$. *Because every 5 points lie in a unique hexad we know that we do not even have to consider method (iv).*

Example 9.3. *What if we have* $\{0,\ 1,\ 2,\ 3,\ 9\}$*? Again we only have to consider methods (iii) and (iv). Adding the points to the* ∞*-picture we get the following.*

6	X	**3**
2	7	4
5	**9**	8

Searching among the possible crosses we can see that these three points never occur in a cross. This is where it sometimes proves helpful to make use of the remark on reflective symmetry as it limits the number of places we can add 2 points. We are left with considering method (iv). We could say that as every 5 points must lie in a unique hexad we can just add ∞ *and know we are done, but let us check that this actually works. All we have to check is that the points we have in bold above is a line. Thankfully we can see that this is a line in picture C defined by* $\circ$*, and so we know that the hexad is* $\{0,\ 1,\ 2,\ 3,\ 9,\ \infty\}$.

10 The MINIMOG and Mathematical Blackjack

Another way to form the hexads of $S(5,6,12)$ is by the MINIMOG which was developed by Conway in [5]. Although the Kitten is easier to work with in later sections, there is a lot to find interesting in the MINIMOG.

10.1 The MINIMOG

To define the MINIMOG we first have to dip our toes in a little coding theory.

Definition 10.1. *The ternary tetracode,* $\mathcal{C}_4 \subset \mathbb{F}_3^4$ *is the linear code whose codewords are given by the following.*

0 0 0 0	0 + + +	0 + + +
+ 0 + -	+ + - 0	+ - 0 +
- 0 - +	- + 0 -	- - + 0

We use the convention that $0 = 0$, $+ = 1$ *and* $- = 2$ *in* $\mathbb{F}_3$.

For a given code word $a\ b\ c\ d$ we can define a function $\phi(x) = ax + b$. And then we find that $\phi(1) = \phi(+) = c$ and $\phi(2) = \phi(-) = d$.

All codewords are of the form:

$$0\ \ a\ \ a\ \ a \qquad\qquad 1\ \ a\ \ b\ \ c \qquad\qquad 2\ \ a\ \ b\ \ c$$

Where a is any of $\{0, 1, 2\}$ and, $a\ b\ c$ is given by any cyclic permutation of 0 1 2.

Remark 10.2. *For* C_4 *we are able to solve the so-called 2-word and 4-word problems.*
Being able to solve the 2-word problem means that given any symbols of a codeword (in the right positions) we can complete this to a unique codeword.
Solving the 4-word problem amounts to being able to correct any codeword for which a single symbol has been changed.
These two properties follow from the fact that the minimum distance between any two codewords in $\mathcal{C}_4$ *is 3. Where we define the (hamming) distance between a pair of codewords to be the the number of positions they differ in. i.e. for two words length* n *we say the distance between them is given by* $d(\underline{x}, \underline{y}) = |\{i \in \{1, \ldots, n\} | x_i \neq y_i\}$.

We use these codewords to find all hexads of $S(5,6,12)$. Consider the shuffle labelling below.

$$\begin{array}{c|cccc|} \cline{2-5} 0 & 6 & 3 & 0 & 9 \\ + & 5 & 2 & 7 & 10 \\ - & 4 & 1 & 8 & 11 \\ \cline{2-5} \end{array}$$

Remark 10.3. *This forms a set of different hexads of $S(5,6,12)$ from those in the section on the Kitten. We use this labelling so as to remain consistent with the labelling of the relevant authors. However we could relabel the MINIMOG as:*

$$\begin{array}{c|cccc|} \cline{2-5} 0 & 0 & 3 & \infty & 2 \\ + & 5 & 9 & 8 & 10 \\ - & 4 & 1 & 6 & 7 \\ \cline{2-5} \end{array}$$

and then we would get the same hexads from both the Kitten and The MINIMOG.

The symbols 0, $+$, $-$ appearing on the left hand side of the MINIMOG will come in handy when forming the "odd man outs" which we discuss later on. We call the first row the 0 row, the second the $+$, and the third the $-$.

In order to find hexads we need to first define cols and tets.
The following shows the four columns.

$$\begin{array}{|cccc|} \hline + & & & \\ + & & & \\ + & & & \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline & + & & \\ & + & & \\ & + & & \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline & & + & \\ & & + & \\ & & + & \\ \hline \end{array} \quad \begin{array}{|cccc|} \hline & & & + \\ & & & + \\ & & & + \\ \hline \end{array}$$

The following are all the tets. Below each tet we indicate in which row the symbols appear.

$$\begin{array}{|cccc|} \hline + & + & + & + \\ & & & \\ & & & \\ \hline \multicolumn{4}{c}{} \\ 0 & 0 & 0 & 0 \end{array} \quad \begin{array}{|cccc|} \hline + & & & \\ & + & + & + \\ & & & \\ \hline \multicolumn{4}{c}{} \\ 0 & + & + & + \end{array} \quad \begin{array}{|cccc|} \hline + & & & \\ & & & \\ & + & + & + \\ \hline \multicolumn{4}{c}{} \\ 0 & - & - & - \end{array}$$

$$\begin{array}{|cccc|} \hline & + & & \\ + & & + & \\ & & & + \\ \hline \multicolumn{4}{c}{} \\ + & 0 & + & - \end{array} \quad \begin{array}{|cccc|} \hline & & & + \\ + & + & & \\ & & + & \\ \hline \multicolumn{4}{c}{} \\ + & + & - & 0 \end{array} \quad \begin{array}{|cccc|} \hline & & + & \\ + & & & + \\ & + & & \\ \hline \multicolumn{4}{c}{} \\ + & - & 0 & + \end{array}$$

$$\begin{array}{|cccc|} \hline & + & & \\ & & & + \\ + & & + & \\ \hline \multicolumn{4}{c}{} \\ - & 0 & - & + \end{array} \quad \begin{array}{|cccc|} \hline & & + & \\ & + & & \\ + & & & + \\ \hline \multicolumn{4}{c}{} \\ - & + & 0 & - \end{array} \quad \begin{array}{|cccc|} \hline & & & + \\ & & + & \\ + & + & & \\ \hline \multicolumn{4}{c}{} \\ - & - & + & 0 \end{array}$$

We note that the symbols below the tets give us all codewords of C_4.

The result of any of the following constructions forms what is referred to as a "signed hexad".

(i) col - col

(ii) tet - tet

(iii) col + col - tet

(iv) col + tet

Example 10.4. *Consider the first column and the tet corresponding to* $- \ + \ 0 \ -$*. Let us use these to form a signed hexad by method (iv).*

$$\boxed{\begin{array}{cccc} + & & & \\ + & & & \\ + & & & \end{array}} + \boxed{\begin{array}{cccc} & & + & \\ & + & & \\ + & & & + \end{array}} = \boxed{\begin{array}{cccc} + & & + & \\ + & + & & \\ - & & & + \end{array}}$$

Referring back up to our shuffle labelling, we arrive at $\{6, 5, -4, 2, 0, 11\}$*. Dropping the signs we get* $\{6, 5, 4, 2, 0, 11\}$ *which is a hexad of* $S(5, 6, 12)$*.*

The word below the tets are made of the "odd man outs". For each columns apply the following rule

(i) If there is only one symbol in that column then the "odd man out" is "0" if it is in the first row, "+" if it is in the second and "-" if it is in the first,

(ii) If there are two symbols in one column then the "odd man out" corresponds to instead taking a single point in the empty position and applying rule (i)

(iii) If there are zero of 3 points in a column then the "odd man out" is labelled as "?"

This gives us a way to see if a set of 6 points we have is a hexad or not. Given any 6 points we can form the word given by the "odd man outs". In any place we get a "?" the choice of symbol is ours. The following theorem is from [6].

Theorem 10.5. *The hexads of* $S(5, 6, 12)$ *are then given by taking all signed hexads formed from the description above and ignoring the signs. Equivalently all hexads of* $S(5, 6, 12)$ *are given by taking collections of 6 points of the shuffle labelling such that the corresponding "odd man outs" form a codeword of* C_4 *with the added condition that we never have the points distributed across the columns as* $\{3, 2, 1, 0\}$ *in any order.*

Note that in the case of the "?" we take in its place, if possible, any symbol which completes the word to a codeword. There can be a maximum of two "?"s for any set of six points. And so this amounts to solving the 2-word problem.

10.2 Mathematical Blackjack

Suppose we have 12 playing cards all of the same suit. All the number cards, the joker, the Ace and the Jack.
The value of a number card is given by its number, the value of a joker is 0, an Ace is 1 and a Jack is 11.
We divide the deck into two and place all cards face up on the table. Take the sum of the first 6, if this is at least 21 then we can play, if not reshuffle the deck and start again.
In turn players take a card from the second lot of 6 and swap it with one of the first, with the condition that this move must lower the total sum of the first 6.
The game continues until a player is forced to make a swap that brings the sum below 21, and so losing the game.
It is easy to spot that to win the game all we have to do is get the sum to be exactly 21, thus forcing our opponent to bring the sum below 21.

Theorem 10.6. *Conway-Ryba Winning Strategy - [20]*
Plot the current game on the block diagram using $*$s. Remove a point to get a set of 5 points. We know that any 5 points can be completed to a unique hexad and so find this hexad and add in the corresponding point to your diagram. If in making this substitution we lower the sum then make this move. Otherwise repeat this strategy for another point.

Suppose we are the first player and we are using this strategy. Problems arise when the cards are dealt and you are already given a hexad! Suppose we remove any of the $*$s, we are left with 5 points of a hexad. By the definition of the Steiner system we cannot replace the removed $*$ by anything other than itself in order to make a hexad. And so according to the strategy there is no winning move!
Part of the reason that this strategy works is that there are exactly 11 hexads with sum 21 and no hexads sum lower than 11, and so once we have transformed the deck into one of these hexads we've won the game.

What is the chance that the first deal gives a hexad? The cards are dealt at random so there are $\binom{12}{6} = 924$ initial starting deals. We know there are 132 hexads and so this happens with probability $\frac{132}{924} = \frac{1}{7}$.

Example 10.7. *Suppose the deck is dealt and we get that the first six cards are* $\{0, 2, 6, 8, 9, 10\}$. *Let us draw this on the shuffle labelling.*

*		*	*
	*		*
		*	

This is not a hexad. Its "odd man out" word is $0 + +-$ which is not a codeword of C_4. So we can proceed with the Conway-Ryba strategy.
We try removing the point 2 and get the following diagram.

*		*	*
			*
		*	

We now look at the possible ways to form a hexad and see if we can form one which contains these five points.
We can use method (ii) with the tetrads corresponding to the codewords $0\ 0\ 0\ 0$ *and* $-\ 0\ -\ +$.

+	+	+	+

$-$

	+		
			+
+		+	

$=$

+		+	+
			-
-		-	

Suppose we try and use this as a move. The above tells us to try and remove the card 2 and replace it with the card 4. However this is an illegal move; it does not lower the sum. Let us try again.
Instead remove the point 6 giving us the following picture.

		*	*
	*		*
		*	

Again we can complete this to a hexad with method (ii) this time using the codewords $+\ -\ 0\ +$ *and* $+\ +\ -\ 0$.

		+	
+			+
	+		

$-$

			+
+	+		
		+	

$=$

		+	-
	-		+
	+	-	

This tells us to try removing the 6 and replacing it with the 1 (or Ace). This is allowed as it lowers the sum from 35 to 30. So we make this move.
Proceeding in this way allows us win the game!

11 M_{12} using Magma

When generating M_{12} in Magma it will prove helpful to generate the conjugate of M_{12} in $Sym(12)$ corresponding to the labelling of the Kitten. This means that as we work through the chamber graph in Section 12, if needs be, we could check what we are doing along the way.

To generate M_{12} we form $H = \langle s, r\rangle$ from [25], as a subgroup of $Sym(12)$ where:

- $s(t) = min(2t, 23 - 2t)$, which gives the permutation $s = (1, 2, 4, 8, 7, 9, 5, 10, 3, 6, 11)$,
- $r(t) = 11 - t$, giving the permutation $r = (0, 11)(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)$.

The permutation s corresponds to the Mongean over shuffle on 12 cards labelled $0, 1, .., 11$ developed by mathematician and French revolutionary Gaspard Monge. And r corresponds to reversing the order of the deck of 12 cards. The Mongean over shuffle is given by transferring cards from one hand to another using the rule that the second card goes on top of the first, the third under the first two, the fourth on top of the first three and so on. 12 perfect Mongean shuffles returns a standard 52 card deck back to its original order.
These actually generate an automorphism group for a different labelling system from the Kitten we discussed in section 9, and so we then conjugate H by $x = (6, 11, 4, 9)(10, 2, 7, 5, 3, 8)(1, 2)$.

We also form a specific Sylow 2 subgroup of M_{12} for B, so that is the stabilizer of the γ_0 we choose in Section 12. For more details see the code in Section 24.

Here we use the set of points $0, 1, 2, \ldots, 10, 11$ to correspond with the labelling used by Monge, and in the Kitten we use $0, 1, 2, \ldots, 10, \infty$ to correspond with the labelling used by Conway and Curtis. When moving between the two we simply identify the point 11 with ∞. To be able to work in MAGMA we must use $1, 2, 3, \ldots, 11, 12$, we do this by moving each point up by one.

Remark 11.1. *Mathieu, Conway and Julius Caesar - There is another visual way to generate M_{12} that was found by Conway in [6]. A Rubik Icosahedron is formed by taking an icosahedron and slicing it in half for each pair of antipodal points. This means that each vertex can be rotated by multiplies of $\frac{2\pi}{5}$ radians, a basic move is when a vertex is rotated by $\frac{2\pi}{5}$ radians. There are 12 of these such basic moves, say $m_1, \ldots, m_{12}$, corresponding to the 12 vertices. With these we can generate M_{12} as:*

$$M_{12} = \langle xy^{-1} \mid x, y \in \{m_1, \ldots, _{12}\} \rangle.$$

These xy^{-1} moves are referred to as "Twist Untwist" moves or as "Crossing the Rubicon."

M_{12} is actually self normalising in $Sym(12)$, and so the number of conjugates of M_{12} in $Sym(12)$ is equal to the number of cosets of M_{12} in $Sym(12)$. $|M_{12}| = 95040$ and $|Sym(12)| = 479001600$, so there are 5040 conjugates of M_{12}. This means that the chances of stumbling on the right conjugate of M_{12} for the Kitten by chance (as I learnt the hard way) is very slim.
Had we used another conjugate of M_{12}, for example had we used H, we still would have found the automorphism group of a Steiner system $S(5, 6, 12)$. But we may have found a Steiner system with a different labelling from the one we wanted.

This poses an interesting question. How many hexads would we need to choose such that we have uniquely defined a choice of Steiner system?
In the case of M_{24} and octads this question has been answered by Curtis in [10]. He showed that it is always possible to choose 8 octads that then uniquely define any choice of $S(5, 8, 24)$. In addition if we choose 7 octads, either these will lie in multiple Steiner systems or could lie in none. As well as being an interesting question to explore it actually gives us a method for checking if an element of $Sym(24)$ lies in a chosen conjugate of M_{24}. We find the eight octads which define the Steiner system for which our choice of conjugate of M_{24} is the automorphism group. We then take a candidate element and apply it to all eight octads. If all their images also lie in the system we have defined then this element of $Sym(24)$ lies in our choice of M_{24}.
A rather simple example of this can be found by again looking at the Fano plane. Given one line we can find 6 different Fano planes all containing this line. If we choose two lines then they must intersect at a shared point. This leaves us with two points left to place, we get two different Fano planes for the two different arrangements so neither one or two lines suffice. Suppose instead we fix three lines, this means there is only one point left to place and one position to place it in, so clearly this uniquely defines the system. As a result in the

case of the Fano Plane we find that three lines suffice.
With more time this might have been an interesting question to explore in terms of how many hexads suffice.

The diameter of the chamber graph of M_{12} is 12, and the disc structure is shown below.

i^{th} Disc	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\Delta_i(\gamma_0)\rvert$	4	8	16	32	64	128	256	384	320	192	64	16
Number of B-orbits	2	2	2	2	3	4	6	6	6	6	3	1

As with chamber graphs of the buildings in Section 6.1 and Section 6.2, the last disc of M_{12} is a single B-orbit. So M_{12} is exhibiting some "building-like" behaviour. As discussed in the previous section it seems interesting to calculate the geodesic closure of a chamber in this last disc and γ_0, this emulates finding an apartment of a building.
We denote the single B-orbit of disc 12 by $\Delta^1_{12}(\gamma_0)$.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\lvert\overline{\{\gamma_0, \Delta^1_{12}(\gamma_0)\}} \cap \Delta_i(\gamma_0)\rvert$	1	4	8	12	16	16	16	16	16	12	8	4	1

Summing along the bottom row shows that $\lvert\overline{\{\gamma_0, \Delta^1_{12}(\gamma_0)\}}\rvert = 130$.

The table below shows the sizes of the B-orbits in each disc and the corresponding DB representative from Magma. This information is included as it is something we will be able to verify by hand in the next section.

Disc	Size	DB-Representative
0	1	1
1	2	43,44
2	4	38,35
3	8	40,42
4	16	2,24
5	16	42,33
	32	34
6	32	21,9,27,26
7	32	29,23,39,7
	64	11,28
8	64	19,4,17,12,16,18
9	32	32,3
	64	13,5,31,6
10	32	8,20,15,30,22,10
11	16	36,25
	32	37
12	16	14

The collapsed adjacency graph of M_{12} is shown below. The labelling convention of B-orbits and edges is the same as for $GL_4(2)$.

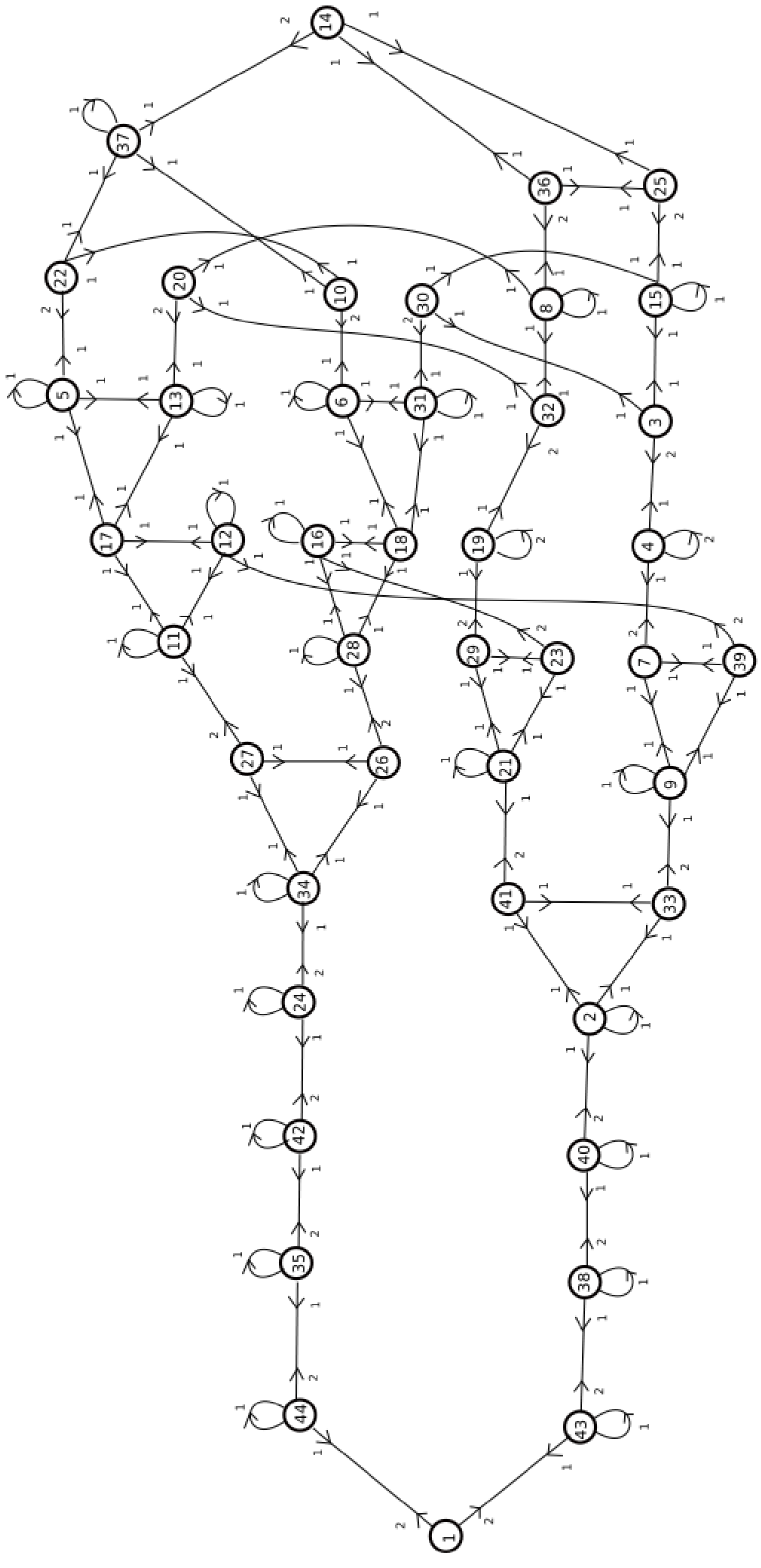


Figure 8: The chamber graph of M_{12} by Magma

12 The Set Up for M_{12} by Hand

In this section we try to form the chamber graph of M_{12} by hand by thinking about things in a combinatorial way. We find our type 1 objects and type 2 objects from the discussion about M_{12} in [24].

A type 1 object is any 4 elements subset of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \times, \infty\}$.

Type 2 objects are a partitioning of the 12 points as $\{X_1 \mid X_2 \mid X_3\}$, such that $|X_i| = 4$, we refer to these as tetrads. However, we cannot consider all such partitioning of the 12 points. Given 4 points there are 4 possible ways to add 2 points so as to make a hexad. This is because we can add any of the 8 remaining points, say x_5, to the 4 to give us 5 points. These must now lie in a unique hexad and so the sixth point, x_6, is uniquely defined. We could have chosen x_6 in place of x_5 and found the same hexad, hence there are $\frac{8}{2} = 4$ hexads which contain the 4 points.
For example to $\{0, 1, 2, 3\}$ we could add $\{4, 6\}$, $\{5, \times\}$, $\{7, 8\}$ or $\{9, \infty\}$. The partitioning of points in a type 2 object respects this, in the sense that given any X_i we induce a pairing on the remaining 8 points. The last two tetrads are given by taking the union of two of these pairs. So if we consider a type 2 object with $\{0, 1, 2, 3\}$ as say X_1 then 4 and 6 must both lie in X_2 or X_3, as for 5 and $\times$ and so on.
For example a type 2 object is given by $\{0, 1, 2, 3 \mid 4, 6, 5, \times \mid 7, 8, 9, \infty\}$. It is also helpful to note that if instead we had chosen $\{4, 6, 5, \times\}$ as X_1 we would have induced the pairing $\{0, 3\}$, $\{1, 2\}$, $\{7, 8\}$, $\{9, \infty\}$ on the remaining points. And for $\{7, 8, 9, \infty\}$ we would have induced $\{0, 3\}$, $\{1, 2\}$, $\{4, 6\}$, $\{5, \times\}$. Note that the pairing of remaining points agrees in all 3 cases.
This partitioning obviously does not hold for all type 2 objects, for example 0 and 3 are not a pair in all such objects. We see this in $\{4, 5, 1, 2 \mid 0, 3, 7, 8 \mid 6, \times, 9, \infty\}$ as $\{4, 5, 1, 2\}$ induces the pairing $\{0, 8\}$, $\{3, 7\}$, $\{6, \times\}$ and $\{9, \infty\}$ on the remaining 8 points. We will need to be careful at each stage when calculating neighbours of a chamber to consult the Kitten for the relevant pairing.

A type 1 object is incident with a type 2 object if the type 1 object is one of the tetrads of the type 2. And so a chamber is given by a type 1 object incident with a type 2. We represent this by underlining the 4 points that make up the type 1 object. For example γ_0 which is the chamber given by $\{0, 1, 2, 3\} \prec \{0, 1, 2, 3 \mid 4, 6, 5, \times \mid 7, 8, 9, \infty\}$ is represented by $\{\underline{0, 1, 2, 3} \mid 4, 6, 5, \times \mid 7, 8, 9, \infty\}$.
We denote the fixed tetrad of γ_0 by $\underline{\gamma_0}$.

Remark 12.1. *The number of chambers for M_{12} is 1485.*

Proof. The number of type 2 objects is $\frac{\binom{12}{4} \cdot \binom{4}{2}}{3!} = 495$.
We have $\binom{12}{4}$ choices for the four points of X_1. Each choice induces a pairing of the remaining points.
There are $\binom{4}{2}$ choices for X_2 by choosing two of the pairs, which then uniquely defines X_3. The ordering of the X_is does not change the type 2 object, and so we divide by three factorial.

We now can choose any of the three X_i to be the type 1 object.
On that account, the number of chambers is $495 \times 3 = 1485$ $\square$

In general two chambers are type 1 neighbours if they share all objects of the same type other than type 1. So here two chambers are type 1 neighbours if they have the same partitioning of the 12 points but a different underlined 4. A type 2 object has 3 subsets of size 4, any of which we can choose to underline. This means there are 3 chambers with the same type 2 object, and so each chamber has 2 neighbours of type 1. For example the type 1 neighbours of γ_0 are given by $\{0,1,2,3 \mid \underline{4,6,5,\times} \mid 7,8,9,\infty\}$ and $\{0,1,2,3 \mid 4,6,5,\times \mid \underline{7,8,9,\infty}\}$.
From now on when doing calculations it makes things clearer to always place the fixed X_i in the first position.

Similarly two chambers are type 2 adjacent if they have the same type 1 object and different type 2. This means the same 4 points are underlined but we can swap the pairs of points about. For a given type 1 object we induce a partitioning of 4 pairs. There are $\binom{4}{2} = 6$ ways of splitting these pairs into X_2 and X_3, as the order of X_2 and X_3 does not matter there are 3 chambers with the same type 1 object. And so each chamber has 2 neighbours of type 2. For example the type 2 neighbours of γ_0 are $\{\underline{0,1,2,3} \mid 4,6,9,\infty \mid 7,8,5,\times\}$ and $\{\underline{0,1,2,3} \mid 4,6,7,8 \mid 5,\times,9,\infty\}$.

To construct the chamber graph we would like to find a way to recognise in which B-orbit a chamber lies in considering some of its properties.
Once we have these properties that define a B-orbit the calculation of the chamber graph becomes much easier. We repeatedly make use Theorem 5.12, in particular that a disc of the chamber graph is the union of B-orbits. And so provided we can show a single element of of a B-orbit lies in say disc i, we can conclude the entire B-orbit must.
We call this method "The Jigsaw Game", as we use it to piece together the B-orbits into discs.

Consider the splitting of the 12 point of the Kitten into the block diagram as shown below:

0	3	4	6	7	8
1	2	5	$\times$	9	∞

a_1	b_1	c_1
a_2	b_2	c_1

=

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$

This shows the choice of X_i for in our fixed chamber γ_0, and also the pairs that each X_i splits into.

Remark 12.2. *That points within each x_i (for $x = a, b, c$, $i = 1, 2$) can be interchanged without changing γ_0, similarly x_1 and x_2 can be interchanged (for $x = a, b, c$), and finally bodily swapping $\mathcal{B}$ and $\mathcal{C}$. We use these observations when discussing how B can act on our block diagram.*

12.1 Types of X_1

Consider the blocks below that show all potential X_1s of a chamber. That is all potential fixed tetrads.

$$Y_1 = \begin{array}{|c|c|c|}\hline \times\ \times & & \\ \hline \times\ \times & & \\ \hline\end{array}$$

$$Y_2 = \begin{array}{|c|c|c|}\hline \times\ \times & \times\ \times & \\ \hline & & \\ \hline\end{array}$$

$$Y_3 = \begin{array}{|c|c|c|}\hline \times\ \times & \times & \\ \hline & \times & \\ \hline\end{array}$$

$$Y_4 = \begin{array}{|c|c|c|}\hline \times\ \times & \times & \times \\ \hline & & \\ \hline\end{array}$$

$$Y_5 = \begin{array}{|c|c|c|}\hline \times & \times\ \times & \\ \hline \times & & \\ \hline\end{array}$$

$$Y_6 = \begin{array}{|c|c|c|}\hline \times & \times & \\ \hline \times & \times & \\ \hline\end{array}$$

$$Y_7 = \begin{array}{|c|c|c|}\hline \times & \times & \times \\ \hline \times & & \\ \hline\end{array}$$

$$Y_8 = \begin{array}{|c|c|c|}\hline \times & \times\ \times & \\ \hline & \times & \\ \hline\end{array}$$

$$Y_9 = \begin{array}{|c|c|c|}\hline \times & \times & \times \\ \hline & \times & \\ \hline\end{array}$$

$$Y_{10} = \begin{array}{|c|c|c|}\hline \times & \times\ \times & \times \\ \hline & & \\ \hline\end{array}$$

$$Y_{11} = \begin{array}{|c|c|c|}\hline & \times\ \times & \\ \hline & \times\ \times & \\ \hline\end{array}$$

$$Y_{12} = \begin{array}{|c|c|c|}\hline & \times\ \times & \\ \hline & & \times\ \times \\ \hline\end{array}$$

$$Y_{13} = \begin{array}{|c|c|c|}\hline & \times\ \times & \times \\ \hline & & \times \\ \hline\end{array}$$

$$Y_{14} = \begin{array}{|c|c|c|}\hline & \times & \times \\ \hline & \times & \times \\ \hline\end{array}$$

$$Y_{15} = \begin{array}{|c|c|c|}\hline & \times & \times \\ \hline & & \times\ \times \\ \hline\end{array}$$

$$Y_{16} = \begin{array}{|c|c|c|}\hline \times\ \times & \times & \\ \hline \times & & \\ \hline\end{array}$$

Remark 12.3. *B stabilizes γ_0 and so by Remark 12.2,B can only act on our block diagram in certain ways. The action of B is given by:*

(i) Interchanging $\mathcal{B}$ and $\mathcal{C}$,

(ii) x_1 interchanged with x_2, for $x = a, b, c$,

(iii) Interchanging two points in x_i for $x = a, b, c$, $i = 1, 2, 3$.

We can see that it is not possible to apply any of the operations (i), (ii) or (iii) as above, to turn Y_i into Y_j for $i \neq j$. This means that with we have a chamber X with $X_1 = Y_i$ and another chamber Z with $Z_1 = Y_j$ for $i \neq j$ then these chambers must lie in different B-orbits.

Note by our previous Remark 12.2, we have that, for example, Y_2 actually represents 8 different X_1s. For more detail see below.

As previously discussed for any choice of X_1 there are 3 chambers.
We can count all chambers with X_1 given by Y_i for some $i = 1 : 16$ to ensure that this will give us all the chambers.

1. Y_1 only represents 1 choice of X_1, as this is an invariant under the action of B.

2. For Y_2 we have 2 choices of the position of the points in $\mathcal{A}$ (both in a_1 or both in a_2), and 4 choices for the remaining points as they can be put in any of b_1, b_2, c_1 or c_2. So we get $2 \times 4 = 8$ choices of X_1 from Y_1.

3. For Y_3 the pair of points in $\mathcal{A}$ can be in two positions, either both in a_1 or a_2. The points in $\mathcal{B}$ can be in any of the 4 positions shown below.

$\{0, 1\}$ $\{0, 2\}$ $\{3, 1\}$ $\{3, 2\}$

But we could also swap $\mathcal{B}$ with $\mathcal{C}$, and so there are $2 \times 4 \times 2 = 16$ choices of X_1 of type Y_3.

4. For Y_4 there are 2 choices for the points in $\mathcal{A}$, namely a_1 and a_2. There are 4 choices for the placement of the point in both $\mathcal{B}$ and $\mathcal{C}$. As a result we get that there are $2 \times 4 \times 4 = 32$ choices for X_1 of type Y_4.

5. In Y_5 the point in a_1 can be in either position as can the point in a_2, so there are 4 choices for the position of the points in $\mathcal{A}$. The 2 points in b_1 could also be in b_2, c_1 or c_2 and still give an X_1 of the same type. Therefore there are $4 \times 4 = 16$ choices of X_1 from Y_5.

6. In the case of Y_6 the point in a_1 can be in either position as can the point in a_2, so there are 4 choices for the position of the points in $\mathcal{A}$. Similarly there are 4 choices of the positions for the points in $\mathcal{B}$, but these points could also be in $\mathcal{C}$ as we do not distinguish between non fixed tetrads. This means that there are $4 \times 4 \times 2 = 32$ choices of X_1 from Y_6.

   Proceeding with the counting in a similar manner we find:

7. $4 \times 4 \times 4 = 64$ choices of X_1 from Y_7.
8. $4 \times 4 \times 2 = 32$ choices of X_1 from Y_8.
9. $4 \times 4 \times 4 \times 2 = 128$ choices of X_1 from Y_9.
10. $4 \times 2 \times 4 \times 2 = 64$ choices of X_1 from Y_{10}.
11. 2 choices of X_1 from Y_{11}.
12. 4 choices of X_1 from Y_{12}.
13. $4 \times 2 \times 2 = 16$ choices of X_1 from Y_{13}.
14. $4 \times 4 = 16$ choices of X_1 from Y_{14}.
15. $4 \times 4 \times 2 = 32$ choices of X_1 from Y_{15}.
16. $4 \times 4 \times 2 = 32$ choices of X_1 from Y_{16}.

Summing these all together gives us 495 choices of X_1, and so $495 \times 3 = 1485$ chambers. This agree with our calculation in Remark 12.1.

12.2 Finding B-orbit Representatives

We make use of the calculation from Section 11 only once in order to know that there are 44 orbits. We now try to find representative elements for each of these orbits by considering how to split each type, Y_i, further. We do this by considering invariants of the other two tetrads X_2 and X_3.

Definition 12.4. *Let $X = \{X_1 \mid X_2 \mid X_3\}$ be a chamber. We say X spreads across $\{a, b, c\}$ tetrads of γ_0 if the elements of X_1 come from a of the tetrads of γ_0, X_2 from b and X_3 from c. (And so clearly $a, b, c \in \{1, 2, 3\}$)*

Note we are taking sets and not sequences here to account for the fact that B can permute tetrads of a chamber.

Definition 12.5. *We talk about $\underline{\gamma_0}$ splitting across a representative chamber $\{X_1 \mid X_2 \mid X_3\}$ as (x_1, x_2, x_3) if $x_i = |\underline{\gamma_0} \cap X_i|$, for $i = 1, 2, 3$. Where $\underline{\gamma_0} = \{0, 1, 2, 3\}$, the type 1 object of γ_0.*

Note that because we can interchange X_2 and X_3 without changing the chamber $\underline{\gamma_0}$ splitting across a representative chamber (x_1, x_2, x_3) is the same as splitting across a representative chamber (x_1, x_3, x_2).

Definition 12.6. *We use the phrase preserved pair to mean that one of a_1, a_2, b_1, b_2, c_1 or c_2 of γ_0 lies in the same X_i for $i = 2, 3$ of a representative chamber.*

Definition 12.7. *We say $X = \{X_1 \mid X_2 \mid X_3\}$ meets $\gamma_0 = \{\gamma_{01} \mid \gamma_{02} \mid \gamma_{03}\}$ (where $\gamma_{01} = \underline{\gamma_0}$) as $[X_2 \cap \gamma_{02},\ X_2 \cap \gamma_{03},\ X_3 \cap \gamma_{02},\ X_3 \cap \gamma_{03}]$.*

Again as we can interchange the two non fixed duads without changing the chamber we note that the entries in $[\cdot,\cdot,\cdot,\cdot]$ are non ordered.

Lemma 12.8. *Considering the action of B as described in Remark 12.3 we can see that the following are B-invariants:*

1. *The way a chamber spreads across the tetrads of γ_0,*
2. *The way γ_0 splits across a chamber,*
3. *How many preserved pairs of $\underline{\gamma_0}$ a chamber has,*
4. *How many preserves pairs of γ_0 not in $\underline{\gamma_0}$ a chamber has,*
5. *Whether or not a chamber contains a preserved tetrad of γ_0,*
6. *The way a chamber meet γ_0.*

We can now use these invariants in order to divide each type, X_i into B-orbits.

Type 1

$Y_{1,1} = \{\underline{0,3,1,2} \mid 4,6,5,\times \mid 7,8,9,\infty\}$
$Y_{1,2} = \{\underline{0,3,1,2} \mid 4,6,7,8 \mid 5,\times,9,\infty\}$

$Y_{1,2}$ cannot lie in the same B-orbit as γ_0, as it spreads across γ_0 as $\{1,2,2\}$ and obviously γ_0 spreads across γ_0 $\{1,1,1\}$.

Type 2

$Y_{2,1} = \{\underline{0,3,4,6} \mid 1,2,9,8 \mid 7,\infty,5,\times\}$
$Y_{2,2} = \{\underline{0,3,4,6} \mid 1,2,5,\times \mid 7,8,9,\infty\}$

$Y_{2,1}$ and $Y_{2,2}$ are representatives for different B-orbits. $Y_{2,2}$ contains a preserved X_i of γ_0 ($\{7,8,9,\infty\}$), $Y_{2,1}$ does not.

Type 3

$Y_{3,1} = \{\underline{0,3,4,5} \mid 9,1,\infty,2 \mid 7,8,6,\times\}$
$Y_{3,2} = \{\underline{0,3,4,5} \mid 9,1,7,8 \mid \infty,2,6,\times\}$

$Y_{3,1}$ and $Y_{3,2}$ are representatives for different B-orbits. The fixed points of γ_0 split across $Y_{3,1}$ as $(2,2,0)$ and across $Y_{3,2}$ as $(2,1,1)$.

Type 4

$Y_{4,1} = \{\underline{0, 3, 4, 7} \mid \times, 1, 9, 2 \mid 5, 8, \infty, 6\}$
$Y_{4,2} = \{\underline{0, 3, 4, 7} \mid \times, 1, 5, 8 \mid 9, 2, \infty, 6\}$
$Y_{4,3} = \{\underline{0, 3, 4, 7} \mid \times, 1, \infty, 6 \mid 9, 2, 5, 8\}$

$Y_{4,1}$ is a representative of a different B-orbit to $Y_{4,2}$ and $Y_{4,3}$, $\underline{\gamma_0}$ splits across $Y_{4,1}$ as $(2, 2, 0)$ whereas $\underline{\gamma_0}$ splits across $Y_{4,2}$ and $Y_{4,3}$ as $(2, 1, 1)$.
$Y_{4,2}$ and $\overline{Y_{4,3}}$ are representatives of different B-orbits, $Y_{4,2}$ preserves some pairs from γ_0 and $Y_{4,3}$ does not. For example $\{5, \times\} \subseteq X_2$ and $\{9, \infty\} \subseteq X_3$ in $Y_{4,2}$.

Type 5

$Y_{5,1} = \{\underline{0, 1, 4, 6} \mid 2, 3, 9, \infty \mid 5, 7, \times, 8\}$
$Y_{5,2} = \{\underline{0, 1, 4, 6} \mid 2, 3, 5, 7 \mid 9, \infty, \times, 8\}$

$Y_{5,1}$ spreads across γ_0 as $\{2, 2, 2\}$, $Y_{5,2}$ spreads across γ_0 as $\{2, 3, 2\}$

Type 6

$Y_{6,1} = \{\underline{0, 1, 4, 5} \mid 9, 3, 2, 8 \mid 6, 7, \times, \infty\}$
$Y_{6,2} = \{\underline{0, 1, 4, 5} \mid 9, 3, 6, 7 \mid 2, 8\times, \infty\}$
$Y_{6,3} = \{\underline{0, 1, 4, 5} \mid 9, 3, \times, \infty \mid 6, 7, 2, 8\}$

$Y_{6,1}$ represents a different B-orbit from $Y_{6,2}$ and $Y_{6,3}$ as $\underline{\gamma_0}$ splits $(2, 2, 0)$, and for the other two we get a splitting of $(2, 1, 1)$.
$Y_{6,2}$ and $Y_{6,3}$ represent different B-orbits: $Y_{6,3}$ contains 2 preserved pairs, $\{9, \infty\} \subseteq X_2$ and $\{8, 7\} \subseteq X_3$, $Y_{6,2}$ contains no preserved pairs.

Type 7

$Y_{7,1} = \{\underline{0, 1, 4, 7} \mid 6, 5, \times, 3 \mid 9, 8, \infty, 2\}$
$Y_{7,2} = \{\underline{0, 1, 4, 7} \mid 6, 5, 9, 8 \mid \times, 3, \infty, 2\}$
$Y_{7,3} = \{\underline{0, 1, 4, 7} \mid 6, 5, \infty, 2 \mid \times, 3, 9, 8\}$
$Y_{7,4} = \{\underline{1, 3, 5, 9} \mid 6, 2, 0, 4 \mid 7, 8, \times, \infty\}$
$Y_{7,5} = \{\underline{1, 3, 5, 9} \mid 6, 2, 7, 8 \mid 0, 4, \times, \infty\}$

We can distinguish $Y_{7,2}$ and $Y_{7,4}$ from the rest: in $Y_{7,2}$ and $Y_{7,4}$ we have that $\underline{\gamma_0}$ splits $(2, 2, 0)$, in the others the split is $(2, 1, 1)$.
$Y_{7,2}$ and $Y_{7,4}$ must represent different B-orbits because $Y_{7,4}$ preserves 2 pairs $\{6, 4\} \subseteq X_2$ and $\{7, 8\} \subseteq X_3$ whereas $Y_{7,2}$ preserves no pairs.
$Y_{7,1}$ contains 2 preserved pairs $\{5, X\} \subseteq X_2$ and $\{9, \infty\} \subseteq X_3$, $Y_{7,5}$ preserves only one pair $\{7, 8\} \subseteq X_2$, and $Y_{7,3}$ preserves none.

Type 8

$Y_{8,1} = \{\underline{0, 4, 6, 5} \mid 1, 7, 9, 2 \mid 3, \times, 8, \infty\}$
$Y_{8,2} = \{\underline{0, 4, 6, 5} \mid 1, 7, 3\times \mid 9, 2, 8, \infty\}$

We can distinguish $Y_{8,1}$ from $Y_{8,2}$ because $Y_{8,1}$ contains a preserved pair of $\underline{\gamma_0}$ (namely $\{1, 2\}$) and $Y_{8,2}$ does not.

Type 9

$Y_{9,1} = \{\underline{6, 7, 9, 2} \mid 1, 4, 0, \infty \mid \times, 5, 3, 8\}$
$Y_{9,2} = \{\underline{6, 7, 9, 2} \mid 3, 8, 0, \infty \mid \times, 5, 1, 4\}$
$Y_{9,3} = \{\underline{6, 8, \infty, 0} \mid 2, 3, 5, 4 \mid 7, 1, 9, \times\}$
$Y_{9,4} = \{\underline{6, 8, \infty, 0} \mid 2, 3, 9, \times \mid 7, 1, 5, 4\}$
$Y_{9,5} = \{\underline{6, 8, \infty, 0} \mid 7, 1, 2, 3 \mid 5, 4, 9, \times\}$
$Y_{9,6} = \{\underline{\times, \infty, 8, 0} \mid 1, 2, 3, 5 \mid 4, 7, 6, 9\}$
$Y_{9,7} = \{\underline{\times, \infty, 8, 0} \mid 1, 2, 4, 7 \mid 3, 5, 6, 9\}$

- $Y_{9,1}$ - γ_0 splits across this $(1, 2, 1)$. It has one preserved pair of the non fixed tetrads of γ_0 and no preserved pair of $\underline{\gamma_0}$.
- $Y_{9,2}$ - γ_0 splits across this $(1, 2, 1)$. It has one preserved pair of the non fixed tetrads of γ_0 and one preserved pair of $\underline{\gamma_0}$.
- $Y_{9,3}$ - γ_0 splits across this $(1, 2, 1)$. It has no preserved pair of the non fixed tetrads of γ_0 and no preserved pair of $\underline{\gamma_0}$. $Y_{9,2}$ spreads across γ_0 as $\{3, 2, 3\}$.
- $Y_{9,4}$ - γ_0 splits across this $(1, 2, 1)$. It has no preserved pair of the non fixed tetrads of γ_0 and no preserved pair of $\underline{\gamma_0}$. $Y_{9,5}$ spreads across γ_0 as $\{3, 3, 3\}$.
- $Y_{9,5}$ - γ_0 splits across this $(1, 3, 0)$. It has one preserved pair of the non fixed tetrads of γ_0 and one preserved pair of $\underline{\gamma_0}$.
- $Y_{9,6}$ - γ_0 splits across this $(1, 3, 0)$. It has one preserved pair of the non fixed tetrads of γ_0 and no preserved pair of $\underline{\gamma_0}$.
- $Y_{9,7}$ - γ_0 splits across this $(1, 3, 0)$. It has no preserved pairs of the non fixed tetrads of γ_0 and no preserved pair of $\underline{\gamma_0}$.

As no two representatives of type Y_9 share all the same invariants we can see that these all must lie in separate B-orbits.

Type 10

$Y_{10,1} = \{\underline{0, 4, 6, 7} \mid 2, 8, 5, 1 \mid 9, \times, 3, \infty\}$
$Y_{10,2} = \{\underline{0, 4, 6, 7} \mid 2, 8, 3, \infty \mid 9, \times, 5, 1\}$
$Y_{10,3} = \{\underline{0, 4, 6, 7} \mid 2, 8, 9, \times \mid 5, 1, 3, \infty\}$

$Y_{10,1}$ is distinct from the other two as it has a shared pair of $\underline{\gamma_0}$ in X_2, namely $\{1,2\}$, and the other two do not.
$Y_{10,2}$ represents a different B-orbit to $Y_{10,3}$, as $Y_{10,2}$ has a shared pair of γ_0 that is not in $\underline{\gamma_0}$, namely $\{5,\times\}$ and $Y_{10,3}$ does not.

Type 11

$Y_{11,1} = \{\underline{4,6,5,\times} \mid 0,3,1,2 \mid 7,8,9,\infty\}$
$Y_{11,2} = \{\underline{4,6,5,\times} \mid 0,3,7,8 \mid 1,2,9,\infty\}$

$\underline{\gamma_0}$ splits across $Y_{11,1}$ as $(0,4,0)$ and $\underline{\gamma_0}$ splits across $Y_{11,2}$ as $(0,2,2)$, and so these represent different B-orbits.

Type 12

$Y_{12,1} = \{\underline{4,6,9,\infty} \mid 2,3,1,0 \mid \times,5,7,8\}$
$Y_{12,2} = \{\underline{4,6,9,\infty} \mid 2,3,\times,5 \mid 1,0,7,8\}$

$\underline{\gamma_0}$ splits across $Y_{12,1}$ as $(0,4,0)$ and across $Y_{12,2}$ as $(0,2,2)$. Hence these two chambers represent separate B-orbits.

Type 13

$Y_{13,1} = \{\underline{4,6,7,9} \mid 2,1,8,\infty \mid 0,\times,3,5\}$
$Y_{13,2} = \{\underline{4,6,7,9} \mid 2,1,0,\times \mid 8,\infty,3,5\}$

$\underline{\gamma_0}$ splits across $Y_{13,1}$ as $(0,2,2)$ and across $Y_{13,2}$ as $(0,3,1)$.

Type 14

$Y_{14,1} = \{\underline{4,\times,7,\infty} \mid 2,9,1,6 \mid 3,5,8,0\}$
$Y_{14,2} = \{\underline{4,\times,7,\infty} \mid 2,9,8,0 \mid 3,5,1,6\}$
$Y_{14,3} = \{\underline{4,\times,7,\infty} \mid 2,9,3,5 \mid 1,6,8,0\}$

$Y_{14,1}$ preserved pairs of $\underline{\gamma_0}$ whereas $Y_{14,2}$ and $Y_{14,3}$ do not.
$Y_{14,2}$ spreads across γ_0 as $\{2,2,2\}$ and $Y_{14,3}$ spreads across γ_0 as $\{2,3,3\}$.

Type 15

$Y_{15,1} = \{\underline{4,7,9,\infty} \mid 3,1,\times,2 \mid 6,8,5,0\}$
$Y_{15,2} = \{\underline{4,7,9,\infty} \mid 3,1,6,8 \mid \times,2,5,0\}$

$\underline{\gamma_0}$ splits across $Y_{15,1}$ as $(0,3,1)$ and as $(0,2,2)$ across $Y_{15,2}$, and so these chambers must represent different B-orbits.

Type 16

$Y_{16,1} = \{\underline{0,3,1,4} \mid 5,9,8,\infty \mid \times,7,6,2\}$
$Y_{16,2} = \{\underline{0,3,1,4} \mid 5,9,\times,7 \mid 8,\infty,6,2\}$

$Y_{16,1}$ meets γ_0 as $[1,3,2,1]$ whereas $Y_{16,2}$ meets γ_0 as $[2,2,1,2]$ and so these must lie in different B-orbits.

This gives 44 representatives and as we know there are 44 B-orbits we know we must have a representative for each of these. We can now construct the chamber graph by playing the Jigsaw Game.
As we shall see for the case of M_{22} we find representative elements for each B-orbit. Given a chamber we then transform it with elements of B until we reach one of our representatives. It is interesting to note that in the case of M_{12} given any chamber we can automatically see which B-orbit it is from. We can do this by just by considering the various ways it intersects γ_0.

Remark 12.9. *This also means that we could easily rephrase the problem for another choice of γ_0, because the same invariants that split chambers into B-orbits would hold. We would just have to change our labelling of points in $a_1, a_2, b_1, b_2, c_1, c_2$.*

13 The Jigsaw Game for M_{12}

We now play "The Jigsaw Game". To do this we make use of Theorem 5.12, that is each B-orbit lies entirely in one disc. And so to show that an entire B-orbit lies in a disc it suffices to show that only one element of the B-orbit lies in that disc.

Recall that we are taking $\gamma_0 = \{\underline{0,1,2,3} \mid 4,6,5,\times \mid 7,8,9,\infty\}$. And so Disc 0 is the single B-orbit formed by this element.

Disc 1

$$\begin{aligned}
Y_{1,1} \ni \{\underline{0,1,2,3} \mid 4,6,5,\times \mid 7,8,9,\infty\} &\xrightarrow{T_1} \{\underline{4,6,5,\times} \mid 0,1,2,3 \mid 7,8,9,\infty\} \in Y_{11,1} \\
&\xrightarrow{T_1} \{\underline{7,8,9,\infty} \mid 4,6,5,\times \mid 0,1,2,3\} \in Y_{11,1} \\
&\xrightarrow{T_2} \{\underline{0,1,2,3} \mid 4,6,7,8 \mid 5,\times,9,\infty\} \in Y_{1,2} \\
&\xrightarrow{T_2} \{\underline{0,1,2,3} \mid 4,6,9,\infty \mid 5,\times,7,8\} \in Y_{1,2}
\end{aligned}$$

This gives us that $\Delta_1(\gamma_0) = \{Y_{11,1}, Y_{1,2}\}$.

Disc 2

We now just take a representative element from each of the B-orbits in Disc 1 rather than looking at every chamber in Disc 1. We would expect to see some B-orbits we have already seen in previous discs at each stage. This is because each B-orbit will have neighbours in the previous disc as well as the possibility of neighbours in the same disc. So when forming the i^{th} disc we need to make sure we only include B-orbits that are neighbours of chambers in the $(i-1)^{th}$ disc that we have not already seen in disc $(i-1)$ or disc $(i-2)$.

$$\begin{aligned} Y_{11,1} \ni \{\underline{4,6,5,\times} \mid 0,1,2,3 \mid 7,8,9,\infty\} &\xrightarrow{T_1} \{\underline{0,1,2,3} \mid 4,6,5,\times \mid 7,8,9,\infty\} \in Y_{1,1} \\ &\xrightarrow{T_1} \{\underline{7,8,9,\infty} \mid 4,6,5,\times \mid 0,1,2,3\} \in Y_{11,1} \\ &\xrightarrow{T_2} \{\underline{4,6,5,\times} \mid 0,3,7,8 \mid 1,2,9,\infty\} \in Y_{11,2} \\ &\xrightarrow{T_2} \{\underline{4,6,5,\times} \mid 0,3,9,\infty \mid 1,2,7,8\} \in Y_{11,2} \end{aligned}$$

$$\begin{aligned} Y_{1,2} \ni \{\underline{0,1,2,3} \mid 4,6,7,8 \mid 5,\times,9,\infty\} &\xrightarrow{T_1} \{\underline{4,6,7,8} \mid 0,1,2,3 \mid 5,\times,9,\infty\} \in Y_{12,1} \\ &\xrightarrow{T_1} \{\underline{5,\times,9,\infty} \mid 4,6,7,8 \mid 0,1,2,3\} \in Y_{12,1} \\ &\xrightarrow{T_2} \{\underline{0,1,2,3} \mid 4,6,5,\times \mid 7,8,9,\infty\} \in Y_{1,1} \\ &\xrightarrow{T_2} \{\underline{0,1,2,3} \mid 4,6,9,\infty \mid 7,8,5,\times\} \in Y_{1,2} \end{aligned}$$

Hence $\Delta_2(\gamma_0) = \{Y_{11,2}, Y_{12,1}\}$.

Disc 3

$$\begin{aligned} Y_{11,2} \ni \{\underline{4,6,5,\times} \mid 0,3,7,8 \mid 1,2,9,\infty\} &\xrightarrow{T_1} \{\underline{0,3,7,8} \mid 4,6,5,\times \mid 1,2,9,\infty\} \in Y_{2,2} \\ &\xrightarrow{T_1} \{\underline{1,2,9,\infty} \mid 0,3,7,8 \mid 4,6,5,\times\} \in Y_{2,2} \\ &\xrightarrow{T_2} \{\underline{4,6,5,\times} \mid 0,1,2,3 \mid 7,8,9,\infty\} \in Y_{11,1} \\ &\xrightarrow{T_2} \{\underline{4,6,5,\times} \mid 0,3,9,\infty \mid 1,2,7,8\} \in Y_{11,2} \end{aligned}$$

$$\begin{aligned} Y_{12,1} \ni \{\underline{4,6,7,8} \mid 0,1,2,3 \mid 5,\times,9,\infty\} &\xrightarrow{T_1} \{\underline{0,1,2,3} \mid 4,6,7,8 \mid 5,\times,9,\infty\} \in Y_{1,2} \\ &\xrightarrow{T_1} \{\underline{5,\times,9,\infty} \mid 4,6,7,8 \mid 0,1,2,3\} \in Y_{12,1} \\ &\xrightarrow{T_2} \{\underline{4,6,7,8} \mid 0,2,5,\times \mid 1,3,9,\infty\} \in Y_{12,2} \\ &\xrightarrow{T_2} \{\underline{4,6,7,8} \mid 0,2,9,\infty \mid 1,3,5,\times\} \in Y_{12,2} \end{aligned}$$

$\Delta_3(\gamma_0) = \{Y_{2,2}, Y_{12,2}\}$.

Disc 4

$$
\begin{aligned}
Y_{2,2} \ni \{\underline{0,3,7,8} \mid 4,6,5,\times \mid 1,2,9,\infty\} &\xrightarrow{T_1} \{\underline{4,6,5,\times} \mid 0,3,7,8 \mid 1,2,9,\infty\} \in Y_{11,2} \\
&\xrightarrow{T_1} \{\underline{1,2,9,\infty} \mid 4,6,5,\times \mid 0,3,7,8\} \in Y_{2,2} \\
&\xrightarrow{T_2} \{\underline{0,3,7,8} \mid 6,\times,9,\infty \mid 4,5,1,2\} \in Y_{2,1} \\
&\xrightarrow{T_2} \{\underline{0,3,7,8} \mid 6,\times,1,2 \mid 4,5,9,\infty\} \in Y_{2,1}
\end{aligned}
$$

$$
\begin{aligned}
Y_{12,2} \ni \{\underline{4,6,7,8} \mid 0,2,5,\times \mid 1,3,9,\infty\} &\xrightarrow{T_1} \{\underline{0,2,5,\times} \mid 4,6,7,8 \mid 1,3,9,\infty\} \in Y_{5,1} \\
&\xrightarrow{T_1} \{\underline{1,3,9,\infty} \mid 0,2,5,\times \mid 4,6,7,8\} \in Y_{5,1} \\
&\xrightarrow{T_2} \{\underline{4,6,7,8} \mid 0,1,2,3 \mid 5,\times,9,\infty\} \in Y_{12,1} \\
&\xrightarrow{T_2} \{\underline{4,6,7,8} \mid 0,2,9,\infty \mid 5,\times,1,3\} \in Y_{12,2}
\end{aligned}
$$

$\Delta_4(\gamma_0) = \{Y_{2,1}, Y_{5,1}\}$.

Disc 5

$$
\begin{aligned}
Y_{2,1} \ni \{\underline{0,3,7,8} \mid 6,\times,9,\infty \mid 4,5,1,2\} &\xrightarrow{T_1} \{\underline{6,\times,9,\infty} \mid 0,3,7,8 \mid 4,5,1,2\} \in Y_{13,1} \\
&\xrightarrow{T_1} \{\underline{4,5,1,2} \mid 6,\times,9,\infty \mid 0,3,7,8\} \in Y_{3,1} \\
&\xrightarrow{T_2} \{\underline{0,3,7,8} \mid 6,\times,4,5 \mid 1,2,9,\infty\} \in Y_{2,2} \\
&\xrightarrow{T_2} \{\underline{0,3,7,8} \mid 6,\times,1,2 \mid 4,5,9,\infty\} \in Y_{2,1}
\end{aligned}
$$

$$
\begin{aligned}
Y_{5,1} \ni \{\underline{0,2,5,\times} \mid 4,6,7,8 \mid 1,3,9,\infty\} &\xrightarrow{T_1} \{\underline{4,6,7,8} \mid 0,2,5,\times \mid 1,3,9,\infty\} \in Y_{12,2} \\
&\xrightarrow{T_1} \{\underline{1,3,9,\infty} \mid 4,6,7,8 \mid 0,2,5,\times\} \in Y_{5,1} \\
&\xrightarrow{T_2} \{\underline{0,2,5,\times} \mid 6,8,1,3 \mid 4,7,9,\infty\} \in Y_{5,2} \\
&\xrightarrow{T_2} \{\underline{0,2,5,\times} \mid 6,8,9,\infty \mid 4,7,1,3\} \in Y_{5,2}
\end{aligned}
$$

$\Delta_5(\gamma_0) = \{Y_{13,1}, Y_{3,1}, Y_{5,2}\}$.

Disc 6

$$
\begin{aligned}
Y_{13,1} \ni \{\underline{6,\times,9,\infty} \mid 0,3,7,8 \mid 4,5,1,2\} &\xrightarrow{T_1} \{\underline{0,3,7,8} \mid 6,\times,9,\infty \mid 4,5,1,2\} \in Y_{2,1} \\
&\xrightarrow{T_1} \{\underline{4,5,1,2} \mid 0,3,7,8 \mid 6,\times,9,\infty\} \in Y_{3,1} \\
&\xrightarrow{T_2} \{\underline{6,\times,9,\infty} \mid 3,7,4,5 \mid 0,8,1,2\} \in Y_{13,2} \\
&\xrightarrow{T_2} \{\underline{6,\times,9,\infty} \mid 3,7,1,2 \mid 0,8,4,5\} \in Y_{13,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,1} \ni \{\underline{4,5,1,2} \mid 0,3,7,8 \mid 6,\times,9,\infty\} &\xrightarrow{T_1} \{\underline{0,3,7,8} \mid 4,5,1,2 \mid 6,\times,9,\infty\} \in Y_{2,1} \\
&\xrightarrow{T_1} \{\underline{6,\times,9,\infty} \mid 0,3,7,8 \mid 4,5,1,2\} \in Y_{13,1} \\
&\xrightarrow{T_2} \{\underline{4,5,1,2} \mid 3,7,6,\times \mid 0,8,9,\infty\} \in Y_{3,2} \\
&\xrightarrow{T_2} \{\underline{4,5,1,2} \mid 3,7,9,\infty \mid 6,\times,0,8\} \in Y_{3,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{5,2} \ni \{\underline{0,2,5,\times} \mid 6,8,1,3 \mid 4,7,9,\infty\} &\xrightarrow{T_1} \{\underline{6,8,1,3} \mid 0,2,5,\times \mid 4,7,9,\infty\} \in Y_{7,4} \\
&\xrightarrow{T_1} \{\underline{4,7,9,\infty} \mid 6,8,1,3 \mid 0,2,5,\times\} \in Y_{7,4} \\
&\xrightarrow{T_2} \{\underline{0,2,5,\times} \mid 6,8,4,7 \mid 1,3,9,\infty\} \in Y_{5,1} \\
&\xrightarrow{T_2} \{\underline{0,2,5,\times} \mid 6,8,9,\infty \mid 1,3,4,7\} \in Y_{15,2}
\end{aligned}
$$

$\Delta_6(\gamma_0) = \{Y_{13,2}, Y_{3,2}, Y_{7,4}, Y_{15,2}\}$.

Disc 7

$$
\begin{aligned}
Y_{13,2} \ni \{\underline{6,\times,9,\infty} \mid 3,7,4,5 \mid 0,8,1,2\} &\xrightarrow{T_1} \{\underline{3,7,4,5} \mid 6,\times,9,\infty \mid 0,8,1,2\} \in Y_{9,6} \\
&\xrightarrow{T_1} \{\underline{0,8,1,2} \mid 3,7,4,5 \mid 6,\times,9,\infty\} \in Y_{16,2} \\
&\xrightarrow{T_2} \{\underline{6,\times,9,\infty} \mid 3,7,0,8 \mid 4,5,1,2\} \in Y_{13,1} \\
&\xrightarrow{T_2} \{\underline{6,\times,9,\infty} \mid 3,7,1,2 \mid 4,5,0,8\} \in Y_{13,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{3,2} \ni \{\underline{4,5,1,2} \mid 3,7,6,\times \mid 0,8,9,\infty\} &\xrightarrow{T_1} \{\underline{3,7,6,\times} \mid 4,5,1,2 \mid 0,8,9,\infty\} \in Y_{9,2} \\
&\xrightarrow{T_1} \{\underline{0,8,9,\infty} \mid 3,7,6,\times \mid 4,5,1,2\} \in Y_{8,1} \\
&\xrightarrow{T_2} \{\underline{4,5,1,2} \mid 3,7,0,8 \mid 6,\times,9,\infty\} \in Y_{3,1} \\
&\xrightarrow{T_2} \{\underline{4,5,1,2} \mid 3,7,9,\infty \mid 6,\times,0,8\} \in Y_{3,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{7,4} \ni \{\underline{6,8,1,3} \mid 0,2,5,\times \mid 4,7,9,\infty\} &\xrightarrow{T_1} \{\underline{0,2,5,\times} \mid 6,8,1,3 \mid 4,7,9,\infty\} \in Y_{5,2} \\
&\xrightarrow{T_1} \{\underline{4,7,9,\infty} \mid 0,2,5,\times \mid 6,8,1,3\} \in Y_{15,2} \\
&\xrightarrow{T_2} \{\underline{6,8,1,3} \mid 2,\times,4,7 \mid 0,5,9,\infty\} \in Y_{7,5} \\
&\xrightarrow{T_2} \{\underline{6,8,1,3} \mid 2,\times,9,\infty \mid 0,5,4,7\} \in Y_{7,5}
\end{aligned}
$$

$$
\begin{aligned}
Y_{15,2} \ni \{\underline{4,7,9,\infty} \mid 0,2,5,\times \mid 6,8,1,3\} &\xrightarrow{T_1} \{\underline{0,2,5,\times} \mid 6,8,1,3 \mid 4,7,9,\infty\} \in Y_{5,2} \\
&\xrightarrow{T_1} \{\underline{6,8,1,3} \mid 0,2,5,\times \mid 4,7,9,\infty\} \in Y_{7,4} \\
&\xrightarrow{T_2} \{\underline{4,7,9,\infty} \mid 2,\times,6,8 \mid 0,5,1,3\} \in Y_{15,1} \\
&\xrightarrow{T_2} \{\underline{4,7,9,\infty} \mid 2,\times,1,3 \mid 0,5,6,8\} \in Y_{15,1}
\end{aligned}
$$

$\Delta_7(\gamma_0) = \{Y_{9,6}, Y_{16,2}, Y_{9,2}, Y_{8,1}, Y_{7,5}, Y_{15,1}\}$.

Disc 8

$$
\begin{aligned}
Y_{9,6} \ni \{\underline{3,7,4,5} \mid 6,\times,9,\infty \mid 0,8,1,2\} &\xrightarrow{T_1} \{\underline{6,\times,9,\infty} \mid 3,7,4,5 \mid 0,8,1,2\} \in Y_{13,2} \\
&\xrightarrow{T_1} \{\underline{0,8,1,2} \mid 6,\times,9,\infty \mid 3,7,4,5\} \in Y_{16,2} \\
&\xrightarrow{T_2} \{\underline{3,7,4,5} \mid \times,\infty,0,8 \mid 6,9,1,2\} \in Y_{9,7} \\
&\xrightarrow{T_2} \{\underline{3,7,4,5} \mid \times,\infty,1,2 \mid 6,9,0,8\} \in Y_{9,7}
\end{aligned}
$$

$$
\begin{aligned}
Y_{16,2} \ni \{\underline{0,8,1,2} \mid 3,7,4,5 \mid 6,\times,9,\infty\} &\xrightarrow{T_1} \{\underline{3,7,4,5} \mid 0,8,1,2 \mid 6,\times,9,\infty\} \in Y_{9,6} \\
&\xrightarrow{T_1} \{\underline{6,\times,9,\infty} \mid 3,7,4,5 \mid 0,8,1,2\} \in Y_{13,2} \\
&\xrightarrow{T_2} \{\underline{0,8,1,2} \mid 4,5,6,9 \mid 3,7,\times,\infty\} \in Y_{16,1} \\
&\xrightarrow{T_2} \{\underline{0,8,1,2} \mid 4,5,\times,\infty \mid 3,7,6,9\} \in Y_{16,1}
\end{aligned}
$$

$$
\begin{aligned}
Y_{9,2} \ni \{\underline{3,7,6,\times} \mid 4,5,1,2 \mid 0,8,9,\infty\} &\xrightarrow{T_1} \{\underline{4,5,1,2} \mid 3,7,6,\times \mid 0,8,9,\infty\} \in Y_{3,2} \\
&\xrightarrow{T_1} \{\underline{0,8,9,\infty} \mid 4,5,1,2 \mid 3,7,6,\times\} \in Y_{8,1} \\
&\xrightarrow{T_2} \{\underline{3,7,6,\times} \mid 5,1,0,8 \mid 4,2,9,\infty\} \in Y_{9,1} \\
&\xrightarrow{T_2} \{\underline{3,7,6,\times} \mid 5,1,9,\infty \mid 4,2,0,8\} \in Y_{9,1}
\end{aligned}
$$

$$Y_{8,1} \ni \{\underline{0,8,9,\infty} \mid 3,7,6,\times \mid 4,5,1,2\} \xrightarrow{T_1} \{\underline{3,7,6,\times} \mid 0,8,9,\infty \mid 4,5,1,2\} \in Y_{9,2}$$
$$\xrightarrow{T_1} \{\underline{4,5,1,2} \mid 3,7,6,\times \mid 0,8,9,\infty\} \in Y_{3,2}$$
$$\xrightarrow{T_2} \{\underline{0,8,9,\infty} \mid 6,\times,4,2 \mid 3,7,5,1\} \in Y_{8,2}$$
$$\xrightarrow{T_2} \{\underline{0,8,9,\infty} \mid 6,\times,5,1 \mid 3,7,4,2\} \in Y_{8,2}$$

$$Y_{7,5} \ni \{\underline{6,8,1,3} \mid 2,\times,4,7 \mid 0,5,9,\infty\} \xrightarrow{T_1} \{\underline{2,\times,4,7} \mid 6,8,1,3 \mid 0,5,9,\infty\} \in Y_{9,1}$$
$$\xrightarrow{T_1} \{\underline{0,5,9,\infty} \mid 2,\times,4,7 \mid 6,8,1,3\} \in Y_{10,3}$$
$$\xrightarrow{T_2} \{\underline{6,8,1,3} \mid 2,\times,0,5 \mid 4,7,9,\infty\} \in Y_{7,4}$$
$$\xrightarrow{T_2} \{\underline{6,8,1,3} \mid 2,\times,9,\infty \mid 4,7,0,5\} \in Y_{7,5}$$

$$Y_{15,1} \ni \{\underline{4,7,9,\infty} \mid 2,\times,6,8 \mid 0,5,1,3\} \xrightarrow{T_1} \{\underline{2,\times,6,8} \mid 4,7,9,\infty \mid 0,5,1,3\} \in Y_{9,5}$$
$$\xrightarrow{T_1} \{\underline{0,5,1,3} \mid 2,\times,6,8 \mid 4,7,9,\infty\} \in Y_{16,1}$$
$$\xrightarrow{T_2} \{\underline{4,7,9,\infty} \mid 2,\times,0,5 \mid 6,8,1,3\} \in Y_{15,2}$$
$$\xrightarrow{T_2} \{\underline{4,7,9,\infty} \mid 2,\times,1,3 \mid 6,8,0,5\} \in Y_{15,1}$$

$\Delta_8(\gamma_0) = \{Y_{9,7}, Y_{8,2}, Y_{10,3}, Y_{9,1}, Y_{16,1}, Y_{9,5}\}$.

Disc 9

$$Y_{9,7} \ni \{\underline{3,7,4,5} \mid \times,\infty,0,8 \mid 6,9,1,2\} \xrightarrow{T_1} \{\underline{\times,\infty,0,8} \mid 3,7,4,5 \mid 6,9,1,2\} \in Y_{9,7}$$
$$\xrightarrow{T_1} \{\underline{6,9,1,2} \mid \times,\infty,0,8 \mid 3,7,4,5\} \in Y_{4,3}$$
$$\xrightarrow{T_2} \{\underline{3,7,4,5} \mid \times,\infty,1,2 \mid 0,8,6,9\} \in Y_{9,4}$$
$$\xrightarrow{T_2} \{\underline{3,7,4,5} \mid \times,\infty,6,9 \mid 0,8,1,2\} \in Y_{9,3}$$

$$Y_{8,2} \ni \{\underline{0,8,9,\infty} \mid 6,\times,4,2 \mid 3,7,5,1\} \xrightarrow{T_1} \{\underline{6,\times,4,2} \mid 0,8,9,\infty \mid 3,7,5,1\} \in Y_{8,2}$$
$$\xrightarrow{T_1} \{\underline{3,7,5,1} \mid 6,\times,4,2 \mid 0,8,9,\infty\} \in Y_{7,1}$$
$$\xrightarrow{T_2} \{\underline{0,8,9,\infty} \mid 6,\times,3,7 \mid 4,2,5,1\} \in Y_{8,1}$$
$$\xrightarrow{T_2} \{\underline{0,8,9,\infty} \mid 6,\times,5,1 \mid 4,2,3,7\} \in Y_{8,2}$$

$$
\begin{aligned}
Y_{10,3} \ni \{\underline{0,5,9,\infty} \mid 6,8,1,3 \mid 2,\times,4,7\} &\xrightarrow{T_1} \{\underline{6,8,1,3} \mid 0,5,9,\infty \mid 2,\times,4,7\} \in Y_{7,5} \\
&\xrightarrow{T_1} \{\underline{2,\times,4,7} \mid 6,8,1,3 \mid 0,5,9,\infty\} \in Y_{9,1} \\
&\xrightarrow{T_2} \{\underline{0,5,9,\infty} \mid 6,3,4,7 \mid 8,1,2,\times\} \in Y_{10,1} \\
&\xrightarrow{T_2} \{\underline{0,5,9,\infty} \mid 8,1,4,7 \mid 6,3,2,\times\} \in Y_{10,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{9,1} \ni \{\underline{2,\times,4,7} \mid 6,8,1,3 \mid 0,5,9,\infty\} &\xrightarrow{T_1} \{\underline{6,8,1,3} \mid 2,\times,4,7 \mid 0,5,9,\infty\} \in Y_{7,5} \\
&\xrightarrow{T_1} \{\underline{0,5,9,\infty} \mid 6,8,1,3 \mid 2,\times,4,7\} \in Y_{10,3} \\
&\xrightarrow{T_2} \{\underline{2,\times,4,7} \mid 8,1,0,5 \mid 9,\infty,6,3\} \in Y_{9,1} \\
&\xrightarrow{T_2} \{\underline{2,\times,4,7} \mid 8,1,9,\infty \mid 0,5,6,3\} \in Y_{9,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{16,1} \ni \{\underline{0,5,1,3} \mid 4,7,9,\infty \mid 2,\times,6,8\} &\xrightarrow{T_1} \{\underline{4,7,9,\infty} \mid 0,5,1,3 \mid 2,\times,6,8\} \in Y_{15,1} \\
&\xrightarrow{T_1} \{\underline{2,\times,6,8} \mid 4,7,9,\infty \mid 0,5,1,3\} \in Y_{9,5} \\
&\xrightarrow{T_2} \{\underline{0,5,1,3} \mid 7,\infty,6,8 \mid 4,9,2,\times\} \in Y_{16,1} \\
&\xrightarrow{T_2} \{\underline{0,5,1,3} \mid 7,\infty,2,\times \mid 4,9,6,8\} \in Y_{16,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{9,5} \ni \{\underline{2,\times,6,8} \mid 4,7,9,\infty \mid 0,5,1,3\} &\xrightarrow{T_1} \{\underline{4,7,9,\infty} \mid 2,\times,6,8 \mid 0,5,1,3\} \in Y_{15,1} \\
&\xrightarrow{T_1} \{\underline{0,5,1,3} \mid 4,7,9,\infty \mid 2,\times,6,8\} \in Y_{16,1} \\
&\xrightarrow{T_2} \{\underline{2,\times,6,8} \mid 4,9,1,3 \mid 0,5,7,\infty\} \in Y_{9,4} \\
&\xrightarrow{T_2} \{\underline{2,\times,6,8} \mid 4,9,0,5 \mid 1,3,7,\infty\} \in Y_{9,3}
\end{aligned}
$$

$\Delta_9(\gamma_0) = \{Y_{4,3}, Y_{7,1}, Y_{10,1}, Y_{10,2}, Y_{9,4}, Y_{9,3}\}$.

Disc 10

$$
\begin{aligned}
Y_{4,3} \ni \{\underline{6,9,1,2} \mid 3,7,4,5 \mid \times,\infty,0,8\} &\xrightarrow{T_1} \{\underline{3,7,4,5} \mid 6,9,1,2 \mid \times,\infty,0,8\} \in Y_{9,7} \\
&\xrightarrow{T_1} \{\underline{\times,\infty,0,8} \mid 3,7,4,5 \mid 6,9,1,2\} \in Y_{9,7} \\
&\xrightarrow{T_2} \{\underline{6,9,1,2} \mid 3,5,0,8 \mid 7,4,\times,\infty\} \in Y_{4,1} \\
&\xrightarrow{T_2} \{\underline{6,9,1,2} \mid 3,5,\times,\infty \mid 7,4,0,8\} \in Y_{4,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{7,1} \ni \{\underline{3,7,5,1} \mid 0,8,9,\infty \mid 6,\times,4,2\} &\xrightarrow{T_1} \{\underline{0,8,9,\infty} \mid 3,7,5,1 \mid 6,\times,4,2\} \in Y_{8,2} \\
&\xrightarrow{T_1} \{\underline{6,\times,4,2} \mid 0,8,9,\infty \mid 3,7,5,1\} \in Y_{8,2} \\
&\xrightarrow{T_2} \{\underline{3,7,5,1} \mid 0,\infty,4,2 \mid 8,9,6,\times\} \in Y_{7,2} \\
&\xrightarrow{T_2} \{\underline{3,7,5,1} \mid 0,\infty,6,\times \mid 8,9,4,2\} \in Y_{7,3}
\end{aligned}
$$

$$
\begin{aligned}
Y_{10,1} \ni \{\underline{0,5,9,\infty} \mid 6,3,4,7 \mid 8,1,2,\times\} &\xrightarrow{T_1} \{\underline{6,3,4,7} \mid 0,5,9,\infty \mid 8,1,2,\times\} \in Y_{10,1} \\
&\xrightarrow{T_1} \{\underline{8,1,2,\times} \mid 6,3,4,7 \mid 0,5,9,\infty\} \in Y_{4,2} \\
&\xrightarrow{T_2} \{\underline{0,5,9,\infty} \mid 6,3,2,\times \mid 8,1,4,7\} \in Y_{10.2} \\
&\xrightarrow{T_2} \{\underline{0,5,9,\infty} \mid 6,3,8,1 \mid 2,\times,4,7\} \in Y_{10.3}
\end{aligned}
$$

$$
\begin{aligned}
Y_{10,2} \ni \{\underline{0,5,9,\infty} \mid 8,1,4,7 \mid 6,3,2,\times\} &\xrightarrow{T_1} \{\underline{8,1,4,7} \mid 0,5,9,\infty \mid 6,3,2,\times\} \in Y_{10,2} \\
&\xrightarrow{T_1} \{\underline{6,3,2,\times} \mid 8,1,4,7 \mid 0,5,9,\infty\} \in Y_{6,3} \\
&\xrightarrow{T_2} \{\underline{0,5,9,\infty} \mid 8,1,6,3 \mid 4,7,2,\times\} \in Y_{10,3} \\
&\xrightarrow{T_2} \{\underline{0,5,9,\infty} \mid 8,1,2,\times \mid 4,7,6,3\} \in Y_{10,1}
\end{aligned}
$$

$$
\begin{aligned}
Y_{9,4} \ni \{\underline{2,\times,6,8} \mid 7,\infty,0,5 \mid 1,3,4,9\} &\xrightarrow{T_1} \{\underline{7,\infty,0,5} \mid 2,\times,6,8 \mid 1,3,4,9\} \in Y_{9,4} \\
&\xrightarrow{T_1} \{\underline{1,3,4,9} \mid 7,\infty,0,5 \mid 2,\times,6,8\} \in Y_{7,3} \\
&\xrightarrow{T_2} \{\underline{2,\times,6,8} \mid 7,\infty,1,3 \mid 4,9,0,5\} \in Y_{9,3} \\
&\xrightarrow{T_2} \{\underline{2,\times,6,8} \mid 7,\infty,4,9 \mid 1,3,0,5\} \in Y_{9,5}
\end{aligned}
$$

$$
\begin{aligned}
Y_{9,3} \ni \{\underline{2,\times,6,8} \mid 7,\infty,1,3 \mid 4,9,0,5\} &\xrightarrow{T_1} \{\underline{7,\infty,1,3} \mid 2,\times,6,8 \mid 4,9,0,5\} \in Y_{6,2} \\
&\xrightarrow{T_1} \{\underline{4,9,0,5} \mid 7,\infty,1,3 \mid 2,\times,6,8\} \in Y_{9,3} \\
&\xrightarrow{T_2} \{\underline{2,\times,6,8} \mid 7,\infty,0,5 \mid 4,9,1,3\} \in Y_{9,4} \\
&\xrightarrow{T_2} \{\underline{2,\times,6,8} \mid 7,\infty,4,9 \mid 0,5,1,3\} \in Y_{9,5}
\end{aligned}
$$

$\Delta_{10}(\gamma_0) = \{Y_{4,1}, Y_{4,2}, Y_{7,2}, Y_{7,3}, Y_{6,3}, Y_{6,2}\}$.

Disc 11

$$
\begin{aligned}
Y_{4,1} \ni \{\underline{6,9,1,2} \mid 3,5,0,8 \mid 7,4,\times,\infty\} &\xrightarrow{T_1} \{\underline{3,5,0,8} \mid 6,9,1,2 \mid 7,4,\times,\infty\} \in Y_{4,1} \\
&\xrightarrow{T_1} \{\underline{7,4,\times,\infty} \mid 3,5,0,8 \mid 6,9,1,2\} \in Y_{14,1} \\
&\xrightarrow{T_2} \{\underline{6,9,1,2} \mid 3,5,7,4 \mid 0,8,\times,\infty\} \in Y_{4,3} \\
&\xrightarrow{T_2} \{\underline{6,9,1,2} \mid 3,5,\times,\infty \mid 0,8,7,4\} \in Y_{4,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{4,2} \ni \{\underline{6,9,1,2} \mid 3,5,\times,\infty \mid 0,8,7,4\} &\xrightarrow{T_1} \{\underline{3,5,\times,\infty} \mid 6,9,1,2 \mid 0,8,7,4\} \in Y_{10,1} \\
&\xrightarrow{T_1} \{\underline{0,8,7,4} \mid 3,5,\times,\infty \mid 6,9,1,2\} \in Y_{10,1} \\
&\xrightarrow{T_2} \{\underline{6,9,1,2} \mid 3,5,0,8 \mid \times,\infty,7,4\} \in Y_{4,1} \\
&\xrightarrow{T_2} \{\underline{6,9,1,2} \mid 3,5,7,4 \mid \times,\infty,0,8\} \in Y_{4,3}
\end{aligned}
$$

$$
\begin{aligned}
Y_{7,2} \ni \{\underline{3,7,5,1} \mid 0,\infty,4,2 \mid 8,9,6,\times\} &\xrightarrow{T_1} \{\underline{0,\infty,4,2} \mid 3,7,5,1 \mid 8,9,6,\times\} \in Y_{7,2} \\
&\xrightarrow{T_1} \{\underline{8,9,6,\times} \mid 0,\infty,4,2 \mid 3,7,5,1\} \in Y_{14,3} \\
&\xrightarrow{T_2} \{\underline{3,7,5,1} \mid 0,\infty,6,\times \mid 8,9,4,2\} \in Y_{7,3} \\
&\xrightarrow{T_2} \{\underline{3,7,5,1} \mid 0,\infty,8,9 \mid 6,\times,4,2\} \in Y_{7,1}
\end{aligned}
$$

$$
\begin{aligned}
Y_{7,3} \ni \{\underline{3,7,5,1} \mid 0,\infty,6,\times \mid 8,9,4,2\} &\xrightarrow{T_1} \{\underline{0,\infty,6,\times} \mid 3,7,5,1 \mid 8,9,4,2\} \in Y_{9,4} \\
&\xrightarrow{T_1} \{\underline{8,9,4,2} \mid 0,\infty,6,\times \mid 3,7,5,1\} \in Y_{9,4} \\
&\xrightarrow{T_2} \{\underline{3,7,5,1} \mid 0,\infty,4,2 \mid 8,9,6,\times\} \in Y_{7,2} \\
&\xrightarrow{T_2} \{\underline{3,7,5,1} \mid 0,\infty,8,9 \mid 4,2,6,\times\} \in Y_{7,1}
\end{aligned}
$$

$$
\begin{aligned}
Y_{6,3} \ni \{\underline{7,\infty,1,3} \mid 4,9,2,6 \mid \times,8,0,5\} &\xrightarrow{T_1} \{\underline{4,9,2,6} \mid 7,\infty,1,3 \mid \times,8,0,5\} \in Y_{10,2} \\
&\xrightarrow{T_1} \{\underline{\times,8,0,5} \mid 4,9,2,6 \mid 7,\infty,1,3\} \in Y_{10,2} \\
&\xrightarrow{T_2} \{\underline{7,\infty,1,3} \mid 4,9,\times,8 \mid 2,6,0,5\} \in Y_{6,1} \\
&\xrightarrow{T_2} \{\underline{7,\infty,1,3} \mid 2,6,\times,8 \mid 4,9,0,5\} \in Y_{6,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{6,2} \ni \{\underline{7,\infty,1,3} \mid 2,\times,6,8 \mid 4,9,0,5\} &\xrightarrow{T_1} \{\underline{2,\times,6,8} \mid 7,\infty,1,3 \mid 4,9,0,5\} \in Y_{9,3} \\
&\xrightarrow{T_1} \{\underline{4,9,0,5} \mid 2,\times,6,8 \mid 7,\infty,1,3\} \in Y_{9,3} \\
&\xrightarrow{T_2} \{\underline{7,\infty,1,3} \mid 2,6,4,9 \mid \times,8,0,5\} \in Y_{6,3} \\
&\xrightarrow{T_2} \{\underline{7,\infty,1,3} \mid \times,8,4,9 \mid 2,6,0,5\} \in Y_{6,1}
\end{aligned}
$$

$\Delta_{11}(\gamma_0) = \{Y_{14,1}, Y_{14,3}, Y_{6,1}\}$.

Disc 12

$$
\begin{aligned}
Y_{14,1} \ni \{\underline{7,4,\times,\infty} \mid 6,9,1,2 \mid 3,5,0,8\} &\xrightarrow{T_1} \{\underline{6,9,1,2} \mid 7,4,\times,\infty \mid 3,5,0,8\} \in Y_{4,1} \\
&\xrightarrow{T_1} \{\underline{3,5,0,8} \mid 6,9,1,2 \mid 7,4,\times,\infty\} \in Y_{4,1} \\
&\xrightarrow{T_2} \{\underline{7,4,\times,\infty} \mid 6,1,3,5 \mid 9,2,0,8\} \in Y_{14,2} \\
&\xrightarrow{T_2} \{\underline{7,4,\times,\infty} \mid 6,1,0,8 \mid 9,2,3,5\} \in Y_{14,3}
\end{aligned}
$$

$$
\begin{aligned}
Y_{14,3} \ni \{\underline{7,4,\times,\infty} \mid 9,2,3,5 \mid 6,1,0,8\} &\xrightarrow{T_1} \{\underline{9,2,3,5} \mid 7,4,\times,\infty \mid 6,1,0,8\} \in Y_{7,2} \\
&\xrightarrow{T_1} \{\underline{6,1,0,8} \mid 9,2,3,5 \mid 7,4,\times,\infty\} \in Y_{7,2} \\
&\xrightarrow{T_2} \{\underline{7,4,\times,\infty} \mid 9,2,6,1 \mid 3,5,0,8\} \in Y_{14,1} \\
&\xrightarrow{T_2} \{\underline{7,4,\times,\infty} \mid 9,2,0,8 \mid 3,5,6,1\} \in Y_{14,2}
\end{aligned}
$$

$$
\begin{aligned}
Y_{6,1} \ni \{\underline{7,\infty,1,3} \mid 4,9,\times,8 \mid 2,6,0,5\} &\xrightarrow{T_1} \{\underline{4,9,\times,8} \mid 7,\infty,1,3 \mid 2,6,0,5\} \in Y_{14,3} \\
&\xrightarrow{T_1} \{\underline{2,6,0,5} \mid 4,9,\times,8 \mid 7,\infty,1,3\} \in Y_{6,1} \\
&\xrightarrow{T_2} \{\underline{7,\infty,1,3} \mid 4,9,2,6 \mid \times,8,0,5\} \in Y_{6,3} \\
&\xrightarrow{T_2} \{\underline{7,\infty,1,3} \mid \times,8,2,6 \mid 4,9,0,5\} \in Y_{6,2}
\end{aligned}
$$

$\Delta_{12}(\gamma_0) = Y_{14,2}$.

We can now draw the chamber graph using all this data. Again we make use of Theorem 5.12 but this time to say that the chamber we have picked to represent the B-orbit is in fact representative in terms of adjacency. For example, if we have shown a chamber $y \in Y_{a,b}$ has a neighbour of type i in $Y_{c,d}$ then we can conclude that every chamber of $Y_{a,b}$ has a neighbour of type i in $Y_{c,d}$.

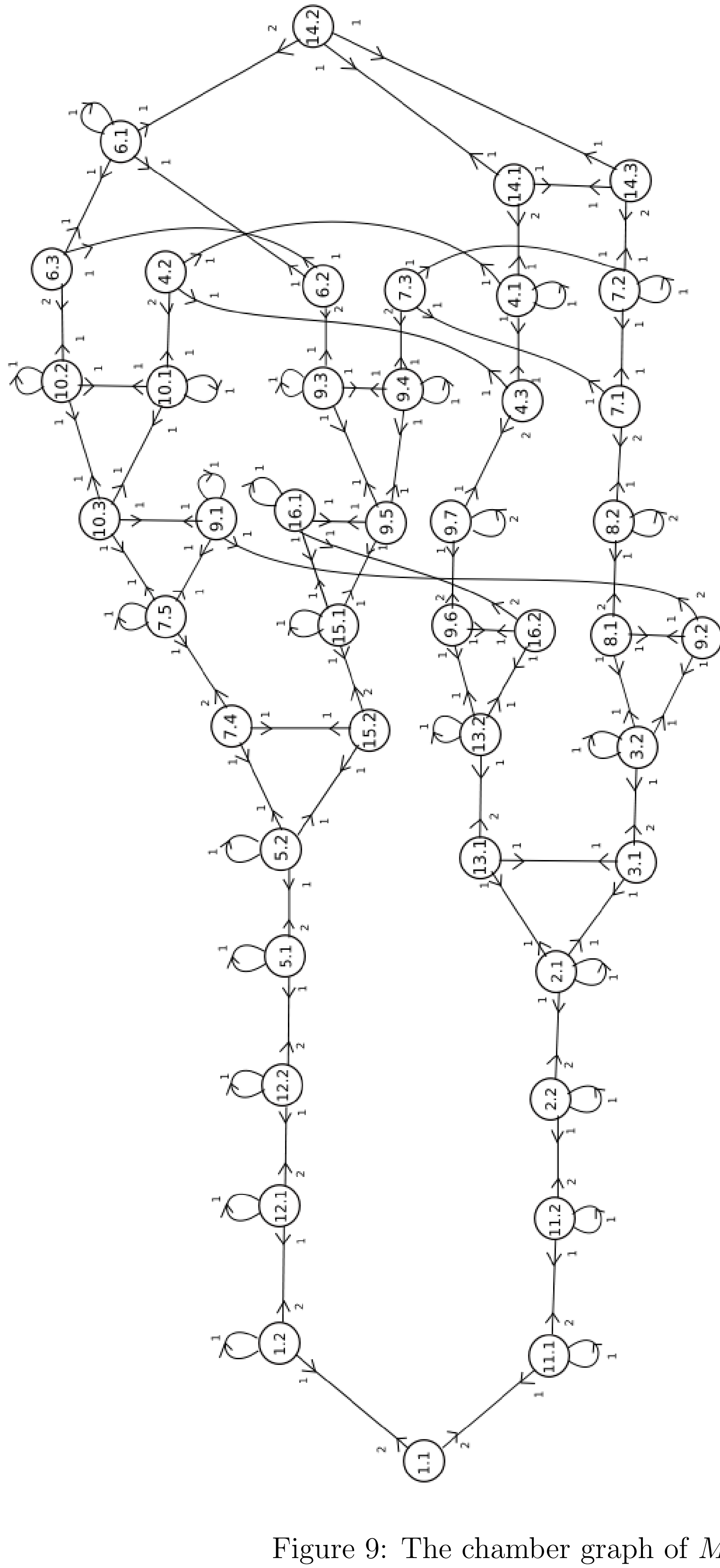


Figure 9: The chamber graph of M_{12} by hand

We can see this is the same diagram as we found using MAGMA but just with a different labelling of orbits.

We can actually get more information from this, in particular we can find the number of chambers in each B-orbit.
Suppose we have the following set up.

Each chamber in A has two neighbours in B, and so the total number of edges is $2|A|$. Each chamber in B has two neighbours in A and so the number of edges is $2|B|$. Equating these we can conclude that $|A| = |B|$.
Each chamber in C has two neighbours in D, and so the total number of edges is $2|C|$. Whereas each chamber in D has only one neighbour in C and so the number of edges is $|D|$. And so we can conclude that $2|C| = |D|$.
We also know that $Y_{1.1}$ contains a single chamber. And so using arguments like those above we calculate the number of chambers in each B-orbit, and so in turn the number of chamber in each disc.
For example $Y_{1.1} \overset{2 \quad 1}{=\!=\!=} Y_{11.1}$. And so $|Y_{11.1}| = 2|Y_{1.1}| = 2$, similarly $|Y_{1.2}| = 2|Y_{1.1}| = 2$. Which gives us that $\Delta_1(\gamma_0) = \{Y_{11.1}, Y_{1.2}\}$ has size 4.

Proceeding as above we get the following:

Disc	Sizes of B-orbits in disc	Size of disc
$\Delta_1(\gamma_0)$	$\lvert Y_{1,2}\rvert = 2$ $\lvert Y_{11,1}\rvert = 2$	4
$\Delta_2(\gamma_0)$	$\lvert Y_{12,1}\rvert = 4$ $\lvert Y_{11,2}\rvert = 4$	8
$\Delta_3(\gamma_0)$	$\lvert Y_{12,2}\rvert = 8$ $\lvert Y_{2,2}\rvert = 8$	16
$\Delta_4(\gamma_0)$	$\lvert Y_{5.1}\rvert = 16$ $\lvert Y_{2,1}\rvert = 16$	32
$\Delta_5(\gamma_0)$	$\lvert Y_{5.2}\rvert = 32$ $\lvert Y_{13.1}\rvert = 16$ $\lvert Y_{3.1}\rvert = 16$	64
$\Delta_6(\gamma_0)$	$\lvert Y_{7.4}\rvert = 32$ $\lvert Y_{15.2}\rvert = 32$ $\lvert Y_{13.2}\rvert = 32$ $\lvert Y_{3.2}\rvert = 32$	128
$\Delta_7(\gamma_0)$	$\lvert Y_{7.5}\rvert = 64$ $\lvert Y_{15.1}\rvert = 64$ $\lvert Y_{9.6}\rvert = 32$ $\lvert Y_{16.2}\rvert = 32$ $\lvert Y_{8.1}\rvert = 32$ $\lvert Y_{9.2}\rvert = 32$	256
$\Delta_8(\gamma_0)$	$\lvert Y_{10.3}\rvert = 64$ $\lvert Y_{9.1}\rvert = 64$ $\lvert Y_{16.1}\rvert = 64$ $\lvert Y_{9.5}\rvert = 64$ $\lvert Y_{9.7}\rvert = 64$ $\lvert Y_{8.2}\rvert = 64$	384
$\Delta_9(\gamma_0)$	$\lvert Y_{10.2}\rvert = 64$ $\lvert Y_{10.1}\rvert = 64$ $\lvert Y_{9.3}\rvert = 64$ $\lvert Y_{9.4}\rvert = 64$ $\lvert Y_{4.3}\rvert = 32$ $\lvert Y_{7.1}\rvert = 32$	320
$\Delta_{10}(\gamma_0)$	$\lvert Y_{6.3}\rvert = 32$ $\lvert Y_{4.2}\rvert = 32$ $\lvert Y_{6.2}\rvert = 32$ $\lvert Y_{7.3}\rvert = 32$ $\lvert Y_{4.1}\rvert = 32$ $\lvert Y_{7.2}\rvert = 32$	192
$\Delta_{11}(\gamma_0)$	$\lvert Y_{6.1}\rvert = 32$ $\lvert Y_{14.1}\rvert = 16$ $\lvert Y_{14.3}\rvert = 16$	64
$\Delta_{12}(\gamma_0)$	$\lvert Y_{14.2}\rvert = 16$	16

Again we can check with our calculations in Section 12, and see that our data for sizes of orbits and discs is the same.

As well as all the information we have already gathered above, we also note that in our drawing of the chamber graph we did not label the type of adjacency but we do have all this information from playing "The Jigsaw Game".

In addition, by Remark 12.9, all of the invariants can be rephrased for any chamber we choose. If we were to be given any two chambers, say X, Z, we could calculate their distance with relative ease. Simply fix the chamber X and now find what B-orbit Z is with respect to $B = Stab_{M_{12}}(X)$. As before we do this by first seeing how the fixed tetrad of Z cuts across X in order to find its type, and then going to the list of that type and seeing specifically which B-orbit it lies in. Then all that remains to be done is to look at the chamber graph drawn above and see which disc Z lies in, if it is disc i then we know the distance between X and Z is i.

Example 13.1. $X = \{\underline{3,5,0,8} \mid 6,9,1,2 \mid 7,4,\times,\infty\}$ *and* $Z = \{\underline{3,7,4,5} \mid 6,\times,9,\infty \mid 0,8,1,2\}$. *We fix* X *and redraw our block as below.*

3	5	9	2	4	7
0	8	6	1	$\times$	∞

We then see that $\underline{Z}$ *cuts as below.*

$\times$	$\times$			$\times$	$\times$

Hence Z *is of type* Y_2*. Looking at the way* Y_2 *splits into* B*-orbits we can see that* Z *lies in* $Y_{2,1}$ *as* Z *does not contain a preserved tetrad of* X*. From the chamber graph we see that* $Y_{2,1}$ *lies in the fourth disc. And so the distance between* X *and* Z *is 4.*

Although initially the calculations were guided by what we knew to be true by results in Magma all the work in this section is done by hand and requires no computation by computer.

14 Maximal Opposite Sets of M_{12}

Now we have found the chamber graph we have a very clear description of the last disc. We can use this to calculate maximal opposite sets. Recall we defined maximal opposite sets to be subsets of our chamber graph of maximal size such that each pair of chambers in the set are at maximal distance.

For M_{12} we know that the diameter 12 and so we want to find a set of chambers all with pairwise distance 12.

Without loss we can take the first chamber we consider to be γ_0. We know all chambers in the last disc are distance 12. As this last disc is a single B-orbit we can choose any of them without loss, say X. (For more detail see Remark 14.1.)
Can we find another chamber that is distance 12 from both γ_0 and X?
First let us calculate the whole of $\Delta_{12}(\gamma_0)$. We know that $\Delta_{12}(\gamma_0)$ is equal to the B-orbit we have called $Y_{14.2}$. All chambers in this orbit spread across γ_0 as $\{2,2,2\}$, they preserve no pairs of $\underline{\gamma_0}$ and their fixed tetrad is of type 14.
We know that there are 16 different choices of fixed tetrad of type 14, so we start with these.
Recall the labelling of the block

0	3	4	6	7	8
1	2	5	$\times$	9	∞

We use this block again to draw out our chambers as it makes it easy to check each chamber we find satisfies the invariants.
For example we represent the chamber $\{4, \times, 7, \infty \mid 2, 9, 8, 0 \mid 3, 5, 1, 6\}$ as below.

$\circ$	$*$	$\bullet$	$*$	$\bullet$	$\circ$
$*$	$\circ$	$*$	$\bullet$	$\circ$	$\bullet$

Where an entry is labelled by a $\bullet$ if the corresponding point in X_1, by a $\circ$ if it is a point of X_2, and $*$ if it is a point of X_3.
Proceeding with this labelling system we find that the 16 choices of fixed tetrad of type 14 gives us the following pictures.

		$\bullet$		$\bullet$	
			$\bullet$		$\bullet$

		$\bullet$		$\bullet$	
			$\bullet$	$\bullet$	

		$\bullet$			$\bullet$
			$\bullet$		$\bullet$

		$\bullet$			$\bullet$
			$\bullet$	$\bullet$	

			$\bullet$	$\bullet$	
			$\bullet$		$\bullet$

			$\bullet$	$\bullet$	
			$\bullet$	$\bullet$	

			$\bullet$		$\bullet$
			$\bullet$		$\bullet$

			$\bullet$		$\bullet$
			$\bullet$	$\bullet$	

		$\bullet$		$\bullet$	
		$\bullet$			$\bullet$

		$\bullet$		$\bullet$	
		$\bullet$		$\bullet$	

		$\bullet$			$\bullet$
		$\bullet$			$\bullet$

		$\bullet$			$\bullet$
		$\bullet$		$\bullet$	

			$\bullet$	$\bullet$	
		$\bullet$			$\bullet$

			$\bullet$	$\bullet$	
		$\bullet$		$\bullet$	

			$\bullet$		$\bullet$
		$\bullet$			$\bullet$

			$\bullet$		$\bullet$
		$\bullet$		$\bullet$	

We know from our earlier discussion that the choice of X_1 for a chamber induces a pairing on on the remaining points. (We find this pairing by noting that the union of a tetrad and

a pair must be a hexad.)
For example the picture:

		$\bullet$		$\bullet$	
			$\bullet$		$\bullet$

,

represents the fixed tetrad $\{4, \times, 7, \infty\}$. These points induce the pairings below:

$$\{\{5,3\},\{2,9\},\{6,1\},\{8,0\}\}.$$

It then remains to divide the pairs between X_2 and X_3 such that the chamber spreads across γ_0 as $\{2,2,2\}$.
For example if we place $\{5,3\}$ in X_2, we then have to choose $\{6,1\}$ to be the other pair in X_2 so that X_2 only spreads across 2 tetrads of γ_0. And then X_3 is given by the remaining two pairs.
Recall that $\{X_1 \mid X_2 \mid X_3\} = \{X_1 \mid X_3 \mid X_2\}$ and so first placing $\{5,3\}$ in X_3 instead of X_2 would not offer us any more possibilities. And this choice of X_1 defines a chamber uniquely.

Similarly for all other choices of X_1 we get a pairing of the remaining points, and the choice of where to put the pairs is uniquely determined by the spreading condition $\{2,2,2\}$.

As a result we get the following 16 chambers lying in $\Delta_{12}(\gamma_0)$.

$\circ$	$\ast$	$\bullet$	$\ast$	$\bullet$	$\circ$
$\ast$	$\circ$	$\ast$	$\bullet$	$\circ$	$\bullet$

$\{\underline{4,\times,7,\infty} \mid 2,9,8,0 \mid 3,5,1,6\}$

$\circ$	$\ast$	$\bullet$	$\circ$	$\bullet$	$\ast$
$\circ$	$\ast$	$\circ$	$\bullet$	$\bullet$	$\ast$

$\{\underline{4,\times,7,9} \mid 5,1,6,0 \mid 2,\infty,3,8\}$

$\circ$	$\ast$	$\bullet$	$\ast$	$\circ$	$\bullet$
$\circ$	$\ast$	$\ast$	$\bullet$	$\circ$	$\bullet$

$\{\underline{4,\times,8,\infty} \mid 9,1,7,0 \mid 2,5,3,6\}$

$\ast$	$\circ$	$\bullet$	$\ast$	$\circ$	$\bullet$
$\circ$	$\ast$	$\ast$	$\bullet$	$\bullet$	$\circ$

$\{\underline{4,\times,8,9} \mid \infty,1,7,3 \mid 0,5,6,2\}$

$\ast$	$\circ$	$\ast$	$\bullet$	$\bullet$	$\circ$
$\ast$	$\circ$	$\ast$	$\bullet$	$\circ$	$\bullet$

$\{\underline{6,\times,7,\infty} \mid 2,8,3,4 \mid 5,0,4,1\}$

$\circ$	$\ast$	$\circ$	$\bullet$	$\bullet$	$\ast$
$\ast$	$\circ$	$\circ$	$\bullet$	$\bullet$	$\ast$

$\{\underline{6,\times,7,9} \mid 4,0,2,5 \mid 8,1,3,\infty\}$

$\circ$	$\ast$	$\ast$	$\bullet$	$\circ$	$\bullet$
$\ast$	$\circ$	$\ast$	$\bullet$	$\circ$	$\bullet$

$\{\underline{6,\times,8,\infty} \mid 9,0,2,7 \mid 3,4,5,1\}$

$\circ$	$\ast$	$\ast$	$\bullet$	$\circ$	$\bullet$
$\circ$	$\ast$	$\ast$	$\bullet$	$\bullet$	$\circ$

$\{\underline{6,\times,8,9} \mid \infty,0,7,1 \mid 5,3,4,2\}$

$\circ$	$\ast$	$\bullet$	$\ast$	$\bullet$	$\circ$
$\circ$	$\ast$	$\bullet$	$\ast$	$\circ$	$\bullet$

$\{\underline{4,5,7,\infty} \mid 9,0,8,1 \mid \times,3,6,2\}$

$\ast$	$\circ$	$\bullet$	$\circ$	$\bullet$	$\ast$
$\circ$	$\ast$	$\bullet$	$\circ$	$\bullet$	$\ast$

$\{\underline{4,5,7,9} \mid \times,1,3,6 \mid \infty,0,2,8\}$

$\ast$	$\circ$	$\bullet$	$\ast$	$\circ$	$\bullet$
$\circ$	$\ast$	$\bullet$	$\ast$	$\circ$	$\bullet$

$\{\underline{4,5,8,\infty} \mid 7,1,3,9 \mid 2,\times,6,0\}$

$\ast$	$\circ$	$\bullet$	$\ast$	$\circ$	$\bullet$
$\ast$	$\circ$	$\bullet$	$\ast$	$\bullet$	$\circ$

$\{\underline{4,5,8,9} \mid 3,\infty,2,7 \mid 0,\times,1,6\}$

$\ast$	$\circ$	$\ast$	$\bullet$	$\bullet$	$\circ$
$\circ$	$\ast$	$\bullet$	$\ast$	$\circ$	$\bullet$

$\{\underline{6,5,7,\infty} \mid 9,1,3,8 \mid 4,2,\times,0\}$

$\circ$	$\ast$	$\ast$	$\bullet$	$\bullet$	$\circ$
$\circ$	$\ast$	$\bullet$	$\ast$	$\bullet$	$\circ$

$\{\underline{6,5,7,9} \mid \infty,1,0,8 \mid 3,4,2,\times\}$

$\ast$	$\circ$	$\ast$	$\bullet$	$\circ$	$\bullet$
$\ast$	$\circ$	$\bullet$	$\ast$	$\circ$	$\bullet$

$\{\underline{6,5,8,\infty} \mid 2,9,3,7 \mid \times,1,4,0\}$

$\circ$	$\ast$	$\ast$	$\bullet$	$\circ$	$\bullet$
$\ast$	$\circ$	$\bullet$	$\ast$	$\bullet$	$\circ$

$\{\underline{6,5,8,9} \mid 2,\infty,7,0 \mid 4,1,3,\times\}$

We now pick the chamber $X = \{\underline{4,\times,7,\infty} \mid 2,9,8,0 \mid 3,5,1,6\}$, and rephrase distance 12 from γ_0 to distance 12 from X.

Any other chamber that is in $\Delta_{12}(X)$ must be type 14.2 with respect to X.
In particular we know that if $Y \in \Delta_{12}(X)$ then Y must be type 14 with respect to Y therefore $\underline{Y} \cap \underline{X} = \emptyset$. Searching among the elements of $\Delta_{12}(\gamma_0)$ for a chamber with this property we can see that the only potential candidate is the one in the bottom right corner, $Y = \{\underline{6, 5, 8, 9} \mid 2, \infty, 7, 0 \mid 4, 1, 3, \times\}$.
We can now check if this in type 14.2 in relation to X.
Its fixed tetrad is type 14 in relation to X. And it spreads across X as $\{2, 2, 2\}$. Hence Y does indeed lie in $\Delta_{12}(X)$.
And so $\Delta_{12}(\gamma_0) \cap \Delta_{12}(X) = \{Y\}$.
$Y \in \Delta_{12}(\gamma_0) \cap \Delta_{12}(X)$ and so is at maximal distance from γ_0 and X. Also γ_0 and X are at maximal distance from one another as X lies in $\Delta_{12}(\gamma_0)$. We need go no further as if we could have any larger maximal opposite set we would need $\Delta_{12}(\gamma_0) \cap \Delta_{12}(X)$ to contain more than one element. And so we can conclude that a maximal opposite set is given by $\{\gamma_0, X, Y\}$. Resultantly the size of a maximal opposite set is 3.

Remark 14.1. *We actually have also shown that M_{12} is transitive on the maximal opposite sets.*

Proof. Suppose we have two such maximal opposite sets. $\{T, U, V\}$ and $\{X, Y, Z\}$.
We can apply an element $g \in M_{12}$ such that $X^g = T$. We then have $\{T, U, V\}$ and $\{T, Y^g, Z^g\}$. The action of M_{12} preserves distance within the chamber graph, so the distance between X and Y is the same as that between $T = X^g$ and Y^g.
Consequently, both U and Y^g are distance 12 from T hence they must lie in $\Delta_{12}(T)$.
This disc forms a single orbit under the action of $B = Stab_{M_{12}}(T)$, for more details see Section 15. Therefore we can find an element $b \in B$ such that $Y^{gb} = U$. Clearly $T^b = T$ for any $b \in B$.
This gives us the two maximal opposite sets $\{T, U, V\}$ and $\{T, U, Z\}$. As we saw, picking the first two elements of the maximal opposite set defines the last one uniquely. Hence $V = Z$.
As a result for any two maximal opposite sets there exists an element of M_{12}, in this case above $gb \in M_{12}$, that maps one to the other.
So M_{12} is indeed transitive on maximal opposite sets. □

15 Verifiying a B-orbit of M_{12}

In the sections above we used Magma for the fact that the number of B-orbits is 44. For interest we now verify by hand that one of our claimed orbits is indeed an orbit.
We have shown that the last disc of the chamber graph of M_{12} is given by all chambers of the type $Y_{14,3}$ and is size 16. We explicitly calculated all of these chambers in Section 14.
We separated chambers in $Y_{14,3}$ from all other chambers by B-invariants, and so we know that $Y_{14,3}$ must be a union of B-orbits. To show that it is a single B-orbit we show that the stabilizer of one of its elements is of size less than or equal to four. Then by the Orbit-Stabilizer theorem we know that this must lie in an orbit of size 16. Hence this must be all of $Y_{14,3}$.
We know from our calculations that $\gamma = \{\underline{7, 4, \times, \infty} \mid 9, 2, 0, 8 \mid 3, 5, 6, 1\}$ is a chamber in the last disc.

Recall the block labelling induced by γ_0.

0 3	4 6	7 8
1 2	5 $\times$	9 ∞

We can represent γ in this block using $\bullet$ to represent the points of X_1, $\circ$ to represent X_2 and $*$ to represent X_3. We get the following correspondence to γ in the block diagram.

$*$ $\circ$	$\bullet$ $\circ$	$\bullet$ $*$
$\circ$ $*$	$\circ$ $\bullet$	$*$ $\bullet$

$B = \langle w, x, y, z \rangle$, where $w = (1,2)(4,\times)(5,6)(7,8)$, $x = (0,2)(1,3)(4,7,6,8)(5,9,\times,\infty)$, $y = (4,\infty,6,9)(5,8,\times,7)$ and $z = (1,2)(4,\times)(5,6)(7,8)$.

Consider restricting the action of B to $\underline{\gamma_0}$. B preserves $\underline{\gamma_0}$ as a set, that is it only maps elements of $\underline{\gamma_0}$ to elements of $\underline{\gamma_0}$. Hence we can think of this restriction as a well defined surjective homomorphism between B and the group we will now call $B_{\underline{\gamma_0}}$.
Let N be the subgroups of B that fixes $\underline{\gamma_0}$ point-wise. N is therefore the kernel of our homomorphism, hence a normal subgroup of B.
The action of the generators x and z restricted to $\underline{\gamma_0}$ are $x' = (0,2)(1,3)$ and $z' = (0,3)$. These generate the order four element $(0,2,3,1)$, and so these two involutions generate $Dih(8)$. Hence the action of B restricted to γ_0 has order at least eight. As a result N can have order at most eight.
Both y and $wxzw = (4,5,6,\times)(7,9,8,\infty)$ fix $\underline{\gamma_0}$ point-wise and so lie in N. If we form $M = \langle y, wxzw \rangle$. We find that:

$$M = \left\{ \begin{array}{llll} (4,5,6,\times)(7,9,8,\infty), & (4,\times,6,5)(7,\infty,8,9), & (4,\infty,6,9)(5,8,\times,7), & (4,9,6,\infty)(5,7,\times,8), \\ (4,8,6,7)(5,9,\times,\infty), & (4,7,6,8)(5,\infty,\times,9), & (4,6)(8,7)(5,\times)(9,\infty), & id \end{array} \right\}$$

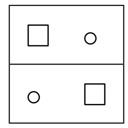
Therefore N must also have order at least eight as it contains a subgroup order eight. We can conclude that $N = M$.
Let S be the subgroup of B that stabilizes γ, our chamber of type $Y_{14,3}$. None of the elements of M stabilize $\underline{\gamma}$, hence $M \cap S = id$.

We want to show that S can have order at most four. M is order eight and so S can be of order at most eight.
S is a subgroup of B, a 2-group, and so if we can show it cannot have order eight then its order must be less than or equal to four.
Suppose for contraction S does have order eight. As a result

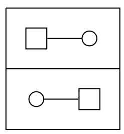
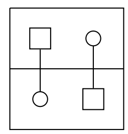
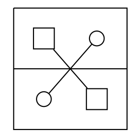
$$|MS| = \frac{|M||S|}{|M \cap S|} = \frac{8 \times 8}{1} = 64 = |B|.$$

Therefore $B = MS$. $N = M$ is normal and so we can factor out M. By the second isomorphism theorem we find $B/M = SM/M \cong S/(S \cap M) \cong S$. Now by the first isomorphism

theorem we find that S is isomorphic to the group $B_{\underline{\gamma_0}}$. We can now think of S as a group order eight acting on the four points of $\underline{\gamma_0}$. Consequently S is a subgroup of $Sym(4)$ order eight, the only such subgroup is $Dih(8)$.
Consider the points of γ lying in the first block as shown below.



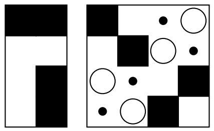
As S stabilizes γ it must preserve these pairs, and so the only possible actions are shown below.

None of which have order four. We reach a contradiction as $Dih(8)$ has elements of order four. So we have shown S must have size strictly less than eight, so S can have order at most four. And so we are done, $Y_{14,3}$ is a B-orbit!

16 The MOG

The MOG was first introduced by Rob Curtis in his paper [9], in 1974. It gives a way to view the Steiner system of M_{24} (and also we shall see it can be used for M_{23} and M_{22}), in a combinatorial way. In particular, it gives all 759 octads of $S(5, 8, 24)$ represented in 35 small diagrams along with a diagram showing the labelling of points. We find the number of octads by noting that any 5 of the 24 points lie in a unique octad, and each octad contains 8 points. There are $\binom{24}{5}$ ways to choose any 5 points and $\binom{8}{5}$ ways of choosing 5 points from 8, and so the number of octads is $\binom{24}{5}/\binom{8}{5} = 759$. Whenever talking about the MOG we label the diagrams 1:35 starting at the top and going left to right, but missing the diagram showing the labelling of the points. The MOG is shown below.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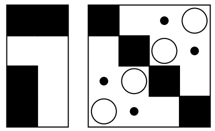
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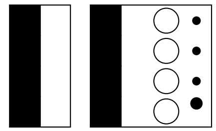
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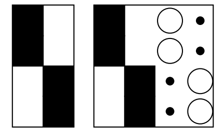
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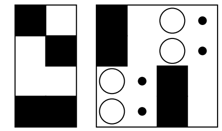
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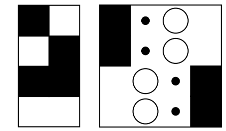
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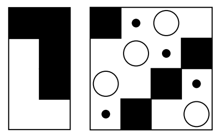
∞	14	17	11	22	19
0	8	4	13	1	9
3	20	16	7	12	5
15	18	10	2	21	6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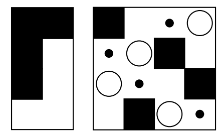
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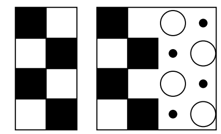
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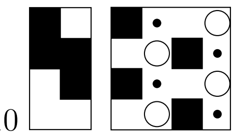
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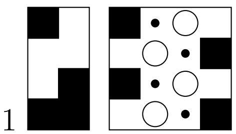
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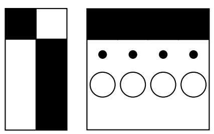
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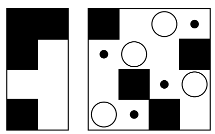
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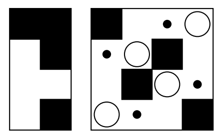
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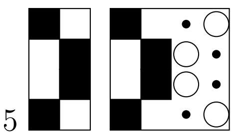
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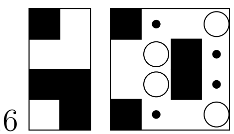
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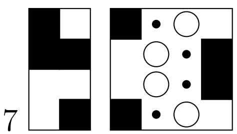
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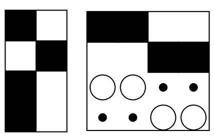
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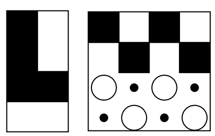
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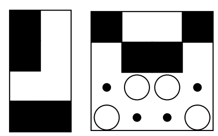
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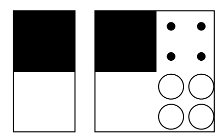
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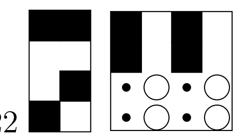
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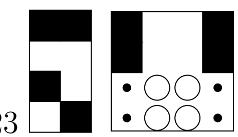
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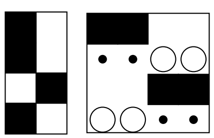
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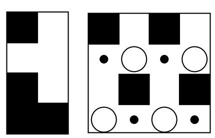
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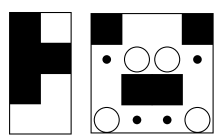
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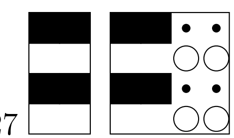
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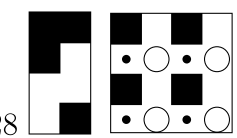
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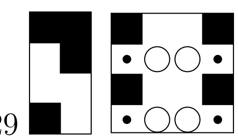
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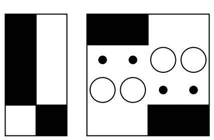
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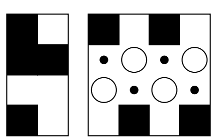
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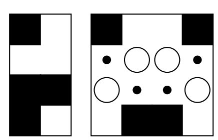
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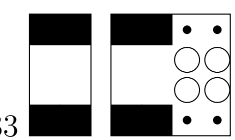
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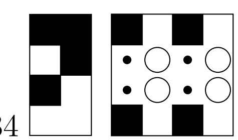
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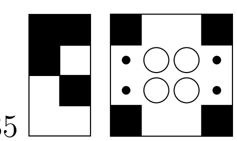
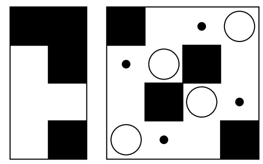
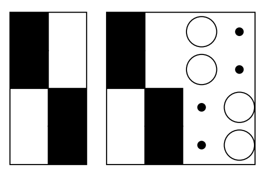

35

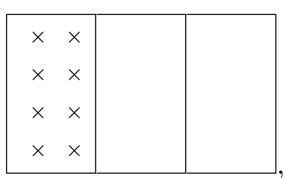
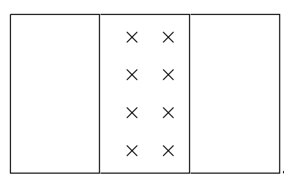
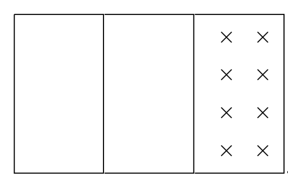
It seems most easy to understand the structure of each diagram through an example. Consider the MOG picture which is 14 in our labelling system, and is shown below.



We call the "brick" on the left the brick tetrad. We can either choose the squares that are black or the squares that are white for the points of the brick tetrad. We then have the "square tetrad" on the right. We can choose either the black squares, white squares, circles or dots for the remaining four points. In addition, we can join the brick tetrad to the start or end of the square tetrad or place it in the middle. This construction gives us an octad. In this case there are two choices for the brick tetrad, four for the square tetrad and three for the placement of the brick tetrad in relation to the square tetrad. This gives us 24 choices from this particular MOG picture. Some MOG pictures offer fewer octads. For example, consider MOG picture 4, shown below.



The brick tetrad has the same structure as both of the bricks that make up the square tetrad and so there are fewer than 24 octads represented by this picture.
The only octads that the MOG does not explicitly show are the so-called heavy bricks,

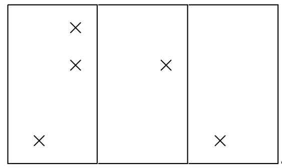
$\Lambda_1 =$ , $\Lambda_2 =$ , $\Lambda_3 =$ .

Remark 16.1. *Any two bricks intersect in only 0, 2, 4 or 8 points. So taking any octad* X *and considering* $X \cap \Lambda_i$ *for* $i = 1, 2, 3$ *we see that each brick can only contain 0, 2 or 4 points. We will use this later when finding octads.*

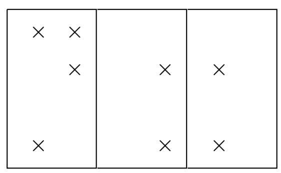
The MOG represents the Steiner system for $S(5, 8, 24)$, any five points lie in a unique octad.

How is The MOG formed? The brick tetrad can be any of the 70 tetrads in that brick. We divide these 70 into 35 groups of size two, with two tetrads together in a group if their union gives the entire brick.
The square tetrads are those tetrads that intersect all rows of the square with the same parity and all columns with the same parity, tetrads satisfying these conditions are called "special tetrads". There are 140 of these, they are grouped into 35 groups of size 4, where two tetrads are together in a group if their union gives an octad.
The MOG shows the way we can relate these sets of 35 groups in such a way that the process described above, of taking a brick tetrad and a square tetrad, forms an octad.

Example 16.2. *Consider the set* $\{8, 13, 14, 15, 21\}$*, can we find which unique octad this lies in? If we draw this in the brick structure of the MOG we have the points below.*



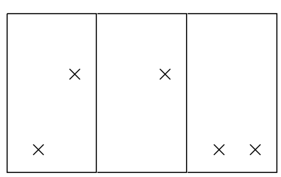
Because a brick can only have 0, 2 or 4 points in we know that the first brick must have 4 points in, and so has to be the brick tetrad. If we search among the MOG we find that the MOG pictures with $\{8, 14, 15\}$ *a subset of its brick tetrad are 4, 16, 19, 29, and 30. We now consider the remaining points* $\{13, 21\}$ *and see if any of the pictures we've selected have these two points represented by the same symbol in the square tetrad. As we would expect from the Steiner system only one of these, namely 29, has these two points also contained in it. The symbol that represents them is the circle. We then add in the remaining point from the brick tetrad and the two other circles in the square tetrad of picture 29. As a result we get the corresponding octad.*



$$\{1, 2, 8, 13, 14, 15, 21, \infty\}$$

Note that because the first brick was the brick tetrad there was no need to permute the bricks given in MOG picture 29.

Example 16.3. *What about when it is not immediately obvious which brick will be the brick tetrad? Consider a slight change to the example above, this time taking the points* $\{6, 8, 13, 15, 21\}$*, giving us the corresponding brick structure below.*



If we consider the first and last brick we see that these points cannot form a special tetrad because they do not satisfy the condition that the points intersect each row with the same parity. As a result we know that the brick tetrad must be the first brick or the last brick of this picture. For convenience we try the third brick as it is an easier pattern to spot in the MOG. We now search the MOG for a brick tetrad that contains these bottom two points. We find the MOG pictures 5, 8, 10, 11, 19, 20, 21, 25, 26, 27, 33, 34 and 35.
We run through this list, now looking for square tetrads for which the points in the first two bricks are a subset.
We find that MOG picture 25 is the only one that works by taking the square tetrad with the circles. All others we could rule out, for example MOG picture 26 is not an option. The

points in the first brick, $\{15, 8\}$, lead us to think that we need to take the circles, but then the point in the second brick, 13, is a dot in this picture. Hence all these points do not lie in the same square tetrad. Similarly we rule all others out apart from 25.
And so the only octad containing these five points is from MOG picture 25, with the brick tetrad in the third position containing the black squares, and the symbol in the square tetrad being the circles. We then add in the remaining points to get the octad below.

				×	
	×		×		
				×	
×		×		×	×

$$\{6, 8, 10, 12, 13, 15, 21, 22\}$$

Definition 16.4. *A sextet is a partitioning of the 24 points in Ω into six tetrads, four element subsets, such that the union of any two is an octad.*

Example 16.5. *Again using MOG picture 14 we get the sextet below.*

1	1	3	5	4	6
2	1	4	6	3	5
2	2	5	3	6	4
2	1	6	4	5	3

The union of 1 and 2 gives Λ_1, the union of 1 or 2 with any of 3,4,5, or 6 gives an octad that arises from MOG picture 14. The union of 3 and 5 or 6 and 4 gives an octad from MOG picture 27. The union of 3 and 6 or 5 and 4 gives an octad from MOG picture 15. The union of 3 and 4 or 5 and 6 gives an octad from MOG picture 4.
Note that the choice of numbering of the tetrads does not matter just which points are grouped together, so for example we could swap the labelling of points 1 and 6 and still have the same sextet.

Definition 16.6. *A trio is a set of three distinct octads.*

Example 16.7. *The following is an example of a trio. The first two octads come from MOG picture 1 and the third from MOG picture 21.*

×	×	×			
			×		
	×				×
	×			×	

			×		
×	×	×			
×				×	
×					×

				×	×
				×	×
		×	×		
		×	×		

These final two definitions will prove useful in section 23.

We can also use the MOG for M_{23} and M_{22}. By Theorem 7.5 we know that M_{23} is the

stabilizer of a point in M_{24}. To use the MOG for M_{23} we fix one point and only consider MOG pictures that contain that one point.
For M_{22} we choose two points to fix and pick the corresponding MOG pictures which contain these two points.
The set of Hexads of M_{22} is then equivalent to only taking octads which contain the two fixed points and then omitting these two points. We will use this when we calculate the chamber graph of M_{22} by hand.

17 M_{22} using Magma

We calculate the chamber graph for M_{22} two different ways, once using code (which can be found in the final section), and then doing the calculations by hand using more combinatorial ideas and making extensive use of the MOG. After we have calculated the chamber graph using MAGMA we then analyse the structure of the chamber graph.

Using the code in MAGMA we can calculate the disc structure and also calculate the sizes of the orbits in each disc. The labelling of the B-orbits is the same as discussed in section 6.

The diameter of the chamber graph of M_{22} is five, and the disc structure is as below.

i^{th} DISC	1	2	3	4	5
$\lvert\Delta_i(\gamma_0)\rvert$	16	56	432	1040	1920
NUMBER OF B-ORBITS	4	6	15	17	17

The last disc, $\Delta_5(\gamma_0)$, is connected. Meaning for any two chambers, X, Y in the last disc we can find a sequence of chambers $Z_1, \ldots, Z_n$ all lying in $\Delta_5(\gamma_0)$, such that $X = Z_1$, $Y = Z_n$ and Z_i is adjacent to Z_{i+1} for all $1 \leq i \leq n-1$.

The table below shows the sizes of the B-orbits in each disc and the corresponding DB representatives from MAGMA.

Disc	Size	DB-Representatives
0	1	1
1	2	2, 44
	4	3
	8	5
2	4	45, 60
	8	4, 52
	16	6, 57
3	8	37, 59
	16	31, 35, 51, 54
	32	17, 26, 28, 30, 46, 53, 58
	64	10, 15
4	8	38, 39
	32	18, 19, 36, 41, 43, 56
	64	27, 29, 47, 50, 55
	128	9, 11, 16, 34
5	64	40, 42, 48, 49
	128	7, 8, 12, 13, 14, 20, 21, 22, 23, 24, 25, 32, 33

Each chamber has adjacency 16, 2 neighbours of type 1, and 14 of type 2. Because of this it would be extremely tedious, and possibly unclear, to draw the collapsed adjacency graph as we did for M_{12}. Instead, we form an adjacency matrix. n is in the i^{th} row and j^{th} column if each chamber in the B-orbit represented by i has n neighbours in the B-orbit represented by j. If i is adjacent to j then clearly j is adjacent to i and so a_{ij} is non zero if and only if a_{ji} is non zero. Therefore the matrix will have symmetry in terms of the non zero entries. We could have that each chamber in the B-orbit i has two neighbours in the B-orbit j, but that each chamber in the B-orbit j has only one neighbour in B-orbit i. Hence a_{ij} might not be equal to a_{ji}. Hence this matrix is symmetric only in terms of shape.

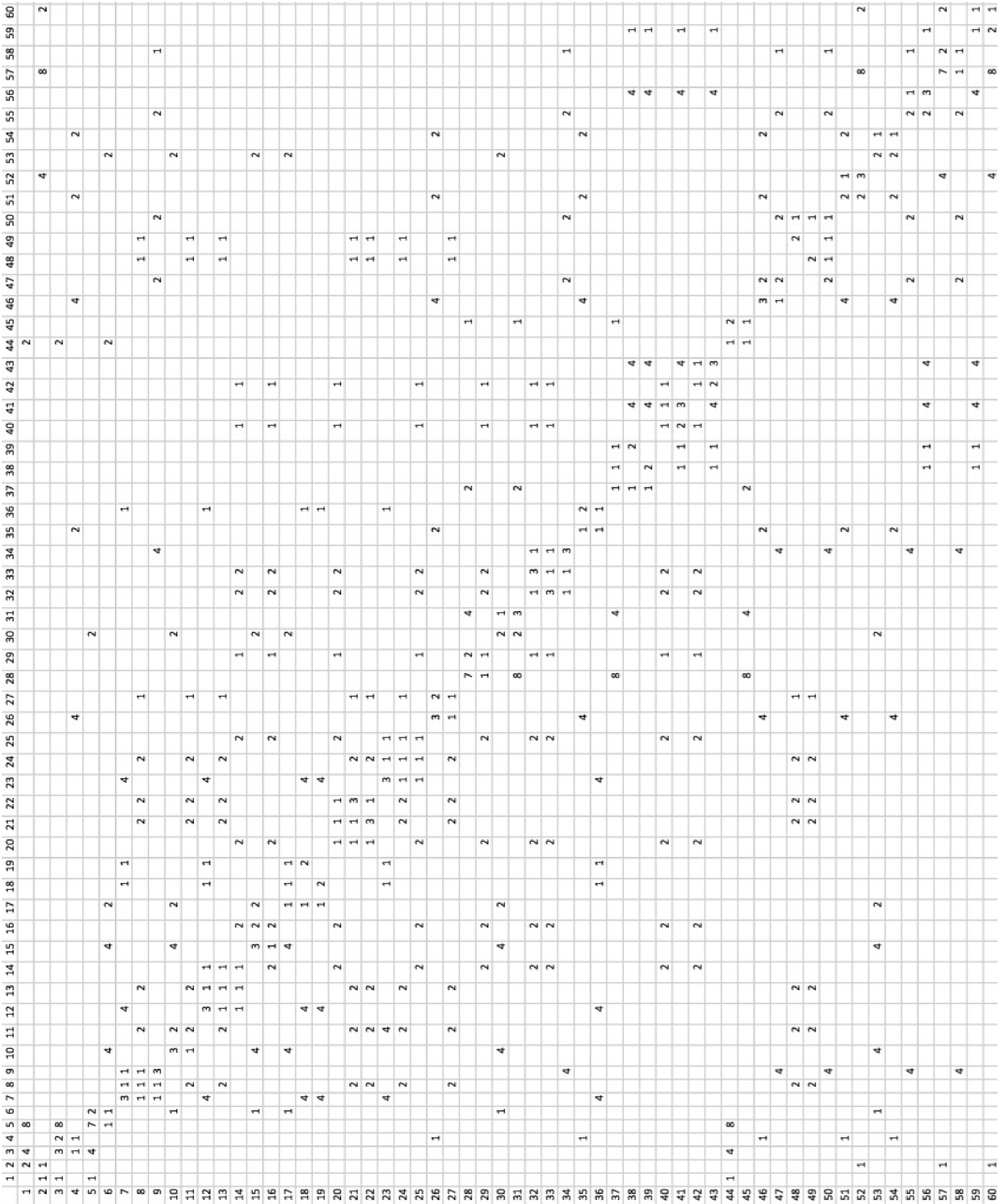


Figure 10: Adjacency Graph of M_{22}

18 The Set Up for M_{22} by Hand

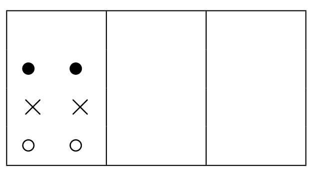
As discussed in Section 16 we form M_{22} as $Stab_{M_{24}}(14) \cap Stab_{M_{24}}(\infty)$. This can be realised in the MOG by taking all pictures that form octads containing these points. We are left with the following MOG pictures; 1, 2, 7, 8, 12, 13, 14, 18, 21, 22, 23, 24, 27, 28, 29, 30, 33, 34, and 35, using the labelling as before. We call this the "Edited MOG".
We find our type 1 objects, and type 2 objects from the discussion in [24] on M_{22}.

Objects of type 1 are duads, which are 2-element sets.
Type 2 objects are triduads. These are 6-element sets formed from the octads in our "Edited MOG" with the points $\{14, \infty\}$ removed, we then divide these six points into three 2-element subsets. A duad is incident with a triduad if the duad is one of the 2-element sets of the triduad.
A chamber is given by a duad, which we call the fixed duad, incident with a triduad. We represent this using the notation $\{\underline{\alpha, \beta} \mid \gamma, \delta \mid \lambda, \mu\}$, where the partitioning shows the duads of the triduad and the underline shows the fixed duad.

Remark 18.1. *The ordering within duads is not important, and neither is the ordering of the duads. So* $\{\underline{\alpha, \beta} \mid \gamma, \delta \mid \lambda, \mu\}$ *is the same chamber as* $\{\underline{\beta, \alpha} \mid \mu, \lambda \mid \delta, \gamma\}$.

It is however, as in the case of M_{12}, clearer to leave the fixed duad at the front.

We can also represent this in an octad of the MOG, for example, $\{\underline{0, 8} \mid 3, 20 \mid 15, 18\}$ can be represented by the diagram below.

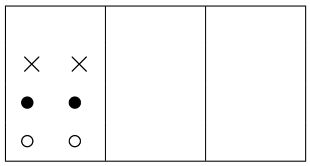
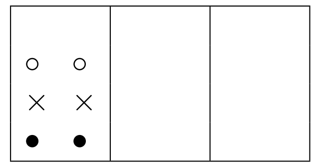


A fixed point is represented by $\bullet$, and the non fixed points are represented by $\times$ and $\circ$ where points share a symbol if they lie in the same duad. The number of type 1 objects is $\binom{22}{2} = 276$.
The number of octads containing the two fixed points, $\{14, \infty\}$, is $\frac{\binom{22}{3}}{\binom{6}{3}} = 77$. Therefore the number of 6-element subsets is 77. There are then $\frac{\binom{6}{2} \times \binom{4}{2}}{3!} = 15$ ways to partition any of the 6-element subsets. Hence there are $77 \times 15 = 1155$ type 2 objects.
To form a chamber we choose any duad of a type 2 object to be the type 1 object, and so there are $1155 \times 3 = 3465$ chambers.

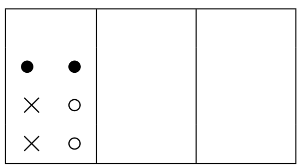
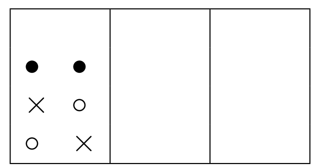
Recall that type 1 neighbours have all spaces the same apart from those of type 1, in this case that translates to having the same triduad, but a different fixed duad. For a given triduad there are 3 choices of duad to have as the fixed. This means for each chamber there are 2 others with the same triduad, and so each chamber has 2 neighbours of type 1.
Type 2 neighbours have all spaces the same apart from those of type 2, and so share the same fixed duad but have different triduads. Using the calculations above but this time fixing the

choice of fixed duad. There are $\frac{\binom{20}{1}}{\binom{4}{1}} = 5$ choices of 6-element sets contains the two points of the fixed duad. There are $\frac{\binom{4}{2}}{2} = 3$ ways to divide the other 4 points of the hexad into duads. Therefore there are 15 chambers with the same fixed duads. Hence each chamber has 14 type 2 neighbours.
Accordingly the valency is 16.
For example using $\gamma_0 = \{\underline{0,8} \mid 3,20 \mid 15,18\}$ we can find disc one by collecting all the chambers which are type 1 and type 2 adjacent.

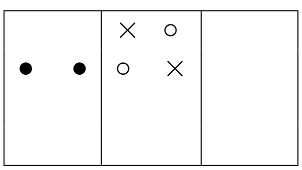
Type 1

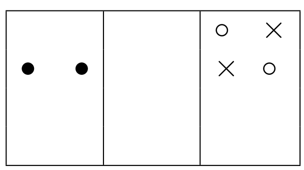
$\{0,8 \mid \underline{3,20} \mid 15,18\} =$  $\{0,8 \mid 3,20 \mid \underline{15,18}\} =$ 

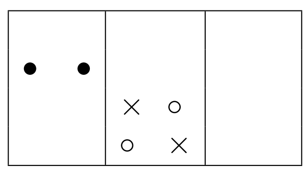
Type 2
We can search among our "Edited MOG" for MOG pictures which contain the fixed duad. We find 21 and Λ_1.
From Λ_1 we get two chambers.

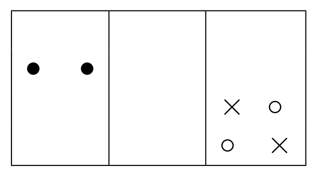
$\{\underline{0,8} \mid 3,15 \mid 20,18\} =$  $\{\underline{0,8} \mid 3,18 \mid 15,20\} =$ 

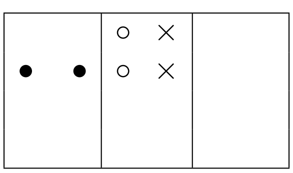
From MOG picture 21 we get the twelve chambers below.

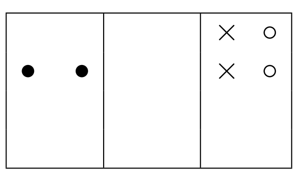

$\{\underline{0,8} \mid 4,11 \mid 17,13\}$

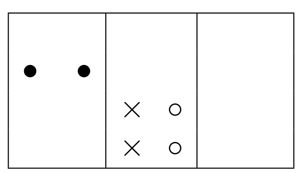

$\{\underline{0,8} \mid 1,19 \mid 22,9\}$

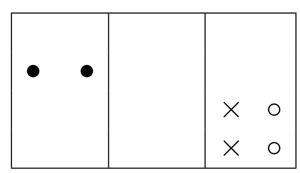

$\{\underline{0,8} \mid 10,7 \mid 16,2\}$

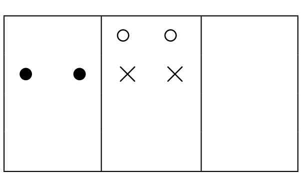

$\{\underline{0,8} \mid 21,5 \mid 12,6\}$

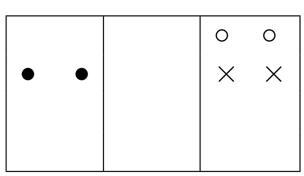

$\{\underline{0,8} \mid 17,4 \mid 11,13\}$

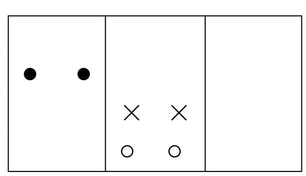

$\{\underline{0,8} \mid 22,1 \mid 19,9\}$

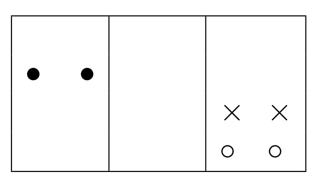

$\{\underline{0,8} \mid 16,19 \mid 7,2\}$


$\{\underline{0,8} \mid 12,21 \mid 5,6\}$


$\{\underline{0,8} \mid 17,11 \mid 4,13\}$


$\{\underline{0,8} \mid 22,19 \mid 1,9\}$


$\{\underline{0,8} \mid 16,7 \mid 10,2\}$


$\{\underline{0,8} \mid 12,5 \mid 21,6\}$

18.1 Finding Generators for B

As with M_{12}, in order to calculate the chamber graph of M_{22} by hand, it is much easier to work with entire B-orbits instead of individual chambers. However, unlike M_{12} it is helpful to have the generators of B. In this section we try to find a set of minimal generators.

We know B is the stabilizer of the fixed chamber $\gamma_0 = \{\underline{0,8} \mid 3,20 \mid 15,18\}$. By Remark 18.1 B must be given by $stab_{M_{22}}(\{0,8\}) \cap stab_{M_{22}}(\{\overline{3,20}\},\{15,18\})$.
The stabilizer of a hexad in M_{22} is $2^4 Alt(6)$ [17], where the 2^4 corresponds to fixing the hexad point-wise and the action on the remaining points. The $Alt(6)$ is the action of M_{22} permuting the points within the hexad. And so $2^4 \leq B \leq 2^4 Alt(6)$.
Dedekind's identity gives us that that $B = 2^4(B \cap Alt(6))$. And so it remains to consider $B \cap Alt(6)$.

Remark 18.2. *The action of B must preserve γ_0, and so the action of B on the points of γ_0 can only be given by combinations of the following:*

1. *interchanging points within the pairs $\{0,8\}$, $\{3,20\}$ and $\{15,18\}$,*
2. *bodily permuting $\{3,20\}$ and $\{15,18\}$.*

Considering $B \cap Alt(6)$, and so by taking only the even cycles of B given by the above, we find that:

$$\begin{aligned} B \cap Alt(6) &= \left\{ \begin{array}{l} id, \ \ (0,8)(3,20), \ \ (0,8)(15,18), \ \ (3,15)(20,18), (3,18)(15,20), \\ (3,20)(15,18), \ \ (3,18,20,15)(0,8), \ \ (3,15,20,18)(0,8) \end{array} \right\} \\ &= \langle a = (3,15,20,18)(0,8),\ b = (0,8)(3,20) \rangle, \quad \text{with } a^4 = 1,\ b^2 = 1,\ bab = a^{-1} \\ &\cong Dih(8) \end{aligned}$$

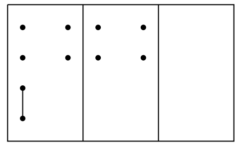
Therefore $B = 2^4 Dih(8) = ED$, where $E = 2^4$ and $D = Dih(8)$.

Remark 18.3. *Here we are using the ATLAS convention, 2^4 represents the elementary abelian group of order 16. This is the group of order 16 in which every non trivial element has order two.*

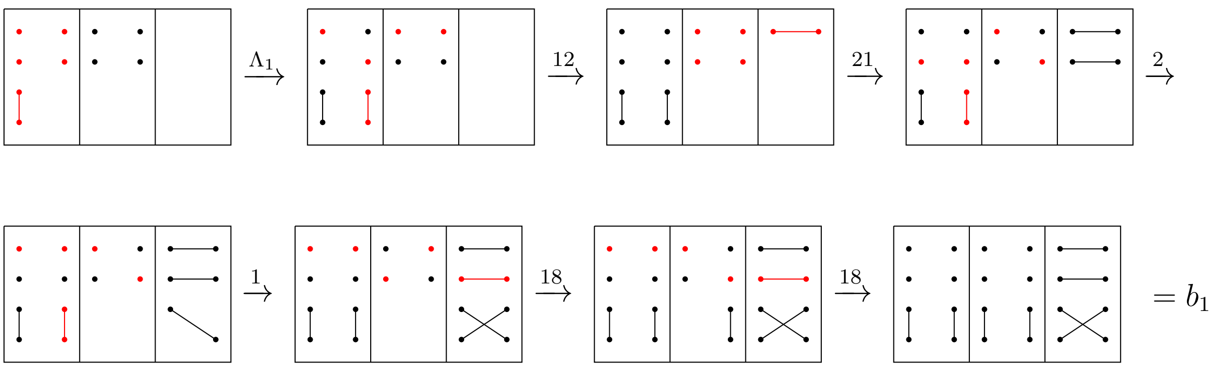
All the generators of E and D will be involutions. We can represent involutions within the structure of an octad, by marking points that are fixed are marked by dots and the points that are interchanged are joined by a line.

Generators for D
D can be generated by two involutions which generate an element of order 4, and so we need to find two different involutions which fix γ_0.
To find such an involution we use a method from [9]. First we find an octad from our "Edited MOG" which we fix, and a further two points that we interchange. We first consider an octad in MOG picture 21 with the two points $\{3,15\}$ being interchanged. Note this choice was guided by the fact that we know the ordering of the non fixed duads in γ_0 does not matter,

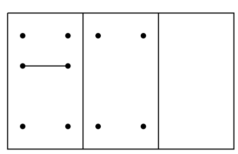
and hence we can see that so far this fixs γ_0.



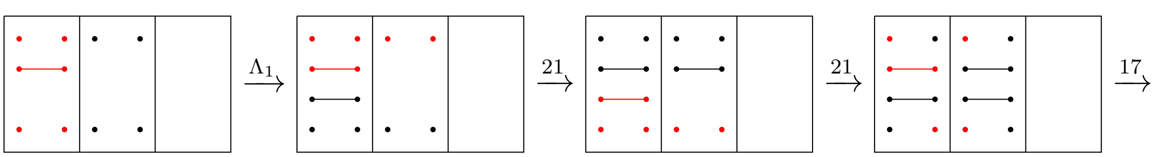
How do we complete this involution? We take the points being interchanged $\{3, 15\}$ and any of the 4 fixed points $\{\infty, 14, 17, 11, 0, 8, 4, 13\}$ and hunt for an octad which contains these 6 points. The remaining 2 points in the octad we find become interchanged. We carry on with this process until we have found involutions for all the remaining points.
The entire process for completing the octad is shown below. At each stage, the six points shown in red are the ones we are considering. The labelled arrow shows which MOG picture we are using that contains the 6 red points.

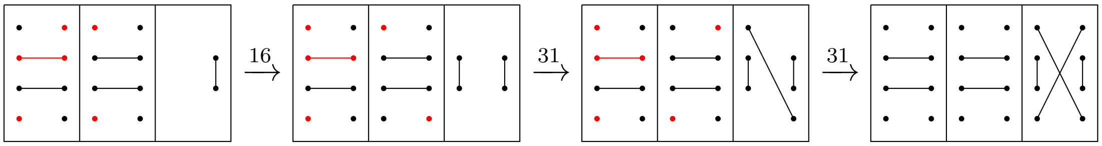


So we reach an involution that fixes $\{0, 8\}$ point-wise and interchanges $\{3, 20\}$ and $\{15, 18\}$. As we hoped this does fix γ_0.
We can also consider an octad from MOG picture 33 and the points $\{0, 8\}$.

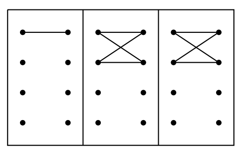


Using the same method as before we can complete this involution as below.

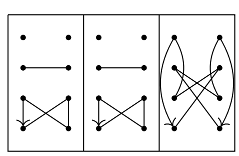


 $= b_2$

b_2 swaps 0 and 8, and also 3, 20, but it fixes the 3 sets $\{0,8\}$, $\{3,20\}$ and $\{15,18\}$ and so fixes γ_0 as we hoped. We know that if these two elements did generate $Dih(8)$ then their product would be order 4. Let us check this. We can rewrite the involutions as permutations and do standard composition.
In permutations we have that the first involution we found has its permutation given by $(3,15)(20,18)(16,10)(7,2)(22,19)(1,9)(12,6)(5,21)$, and the second is given by $(0,8)(3,20)(4,13)(16,7)(22,6)(19,2)(1,12)(9,5)$. Taking their composition we get $(3,15,20,18)(16,10,7,2)(22,21,9,12)(19,6,1,5)(0,8)(4,13)$ which, thankfully is an element of order 4.
In order to represent this in octad structure, we are required to put arrows on lines that represent permutations of order greater than 2. Otherwise, there would be confusion as to which permutation the diagram below represents.



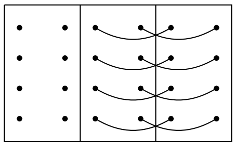
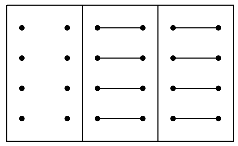
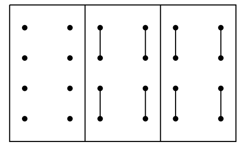
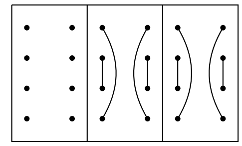
It could actually represent any of the 4 permutations $(17,13,4,11)(22,9,1,19)(\infty,14)$, $(17,13,4,11)(22,19,1,9)(\infty,14)$, $(17,11,4,13)(22,9,1,19)(\infty,14)$, and $(17,11,4,13)(22,19,1,9)(\infty,14)$.
It suffices to put an arrowhead on a single line of an element of order greater than 2, this then uniquely determines the permutation.
As a result to represent the permutation $(3,15,20,18)(16,10,7,2)(22,21,9,12)(19,6,1,5)(0,8)(4,13)$ we use the following diagram.

$b_1 b_2 =$ 

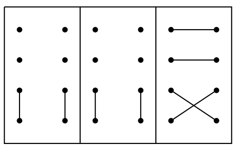
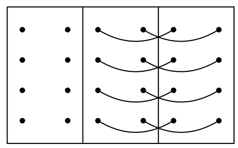
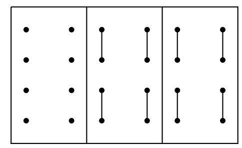
After a little practice, this can just be done by looking at the two diagrams without writing out the permutations.

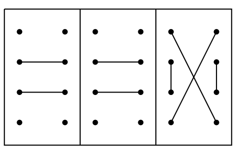
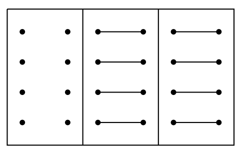
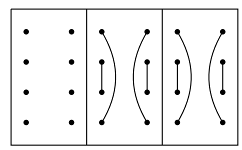
Generators of E
To find the generators of E we note that all elements of E need to fix γ_0 point-wise, and as these should lie in M_{22} they must also fix $\{\infty,14\}$, therefore they must actually fix Λ_1 point-wise. Using the process to complete involutions as before we know we must choose Λ_1 as the octad and any other two points of Ω to interchange. We know that E should have

4 generators. Once we have calculated the first, it remains to again take the octad Λ_1 and choose two points to interchange that are not interchanged in any of our previous generators for E. This ensures we do not get any duplication. Using this process we find that the generators for $E = 2^4$ are given by:

$b_3 =$  $b_4 =$  $b_5 =$  $b_6 =$ 

And so the generators of B are given by:

$b_1 =$  $b_3 =$  $b_5 =$ 

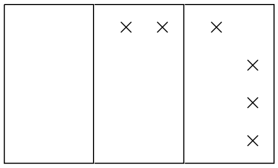
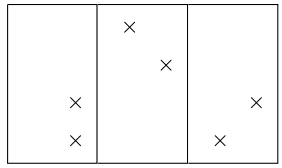
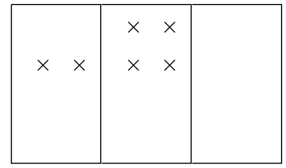
$b_2 =$  $b_4 =$  $b_6 =$ 

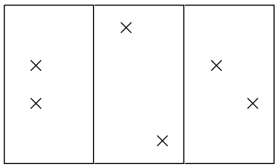
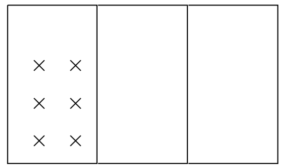
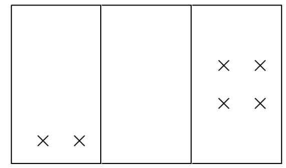
19 Hexad Orbits of M_{22}

19.1 Underlying Structure

As with M_{12} we want a way to recognise which B-orbit a given chamber lies in and then play “The Jigsaw Game”.
This time rather than considering properties that define which orbit a chamber lies in we find a representative chamber for each orbit. We first divide the chambers into hexad orbits and then divide the hexad orbits into chamber orbits. The hexad of a chamber is given by the points of the chamber itself without the splitting of duads specified.

We claim that the following hexads are representatives for the hexad orbits.

$X_1 =$  $X_2 =$  $X_3 =$ 

$X_4 =$  $X_5 =$  $X_6 =$ 

First we show that each of these hexads must lie in different hexad orbits. To do this we find a set of invariants that no two of $X_1, \ldots, X_6$ share.
The invariants we are going to use are:

(i) Size of the intersection with our fixed chamber γ_0,

(ii) Whether or not the intersection of X and γ_0 is a duad of γ_0,

(iii) If $X_i \cap \gamma_0$ is a duad of γ_0, then is it the fixed duad of γ_0, denoted $\underline{\gamma_0}$,

(iv) If $X_i \cap \gamma_0$ is not a duad of γ_0, then how many of the intersection points are ones that lie in the fixed duad of γ_0.

Using Remark 18.2, to see how B can act on the points of γ_0, we can see that these are indeed invariants under B.

Now it remains to show that each of the underlying structures are separated from one another by at least one invariant.

Structure	$\lvert X_i \cap \gamma_0 \rvert$	$X \cap \gamma_0 =$ duad?	$X \cap \gamma_0 =$ fixed duad?	$\lvert X_i \cap \underline{\gamma_0} \rvert$
X_1	0	no	no	0
X_2	2	no	no	0
X_3	2	yes	yes	2
X_4	2	no	no	1
X_5	6	yes	yes	6
X_6	2	yes	no	2

Because no two rows of the table are the same, we can conclude that any two chambers with different underlying structures cannot lie in the same B-orbit.
Hence we know each of these underlying structures represents a union of B-orbits. Let us show that this list of underlying structures is exhaustive, we do this by counting the number of chambers it gives.
We can see which MOG pictures correspond to each of the underlying structures.

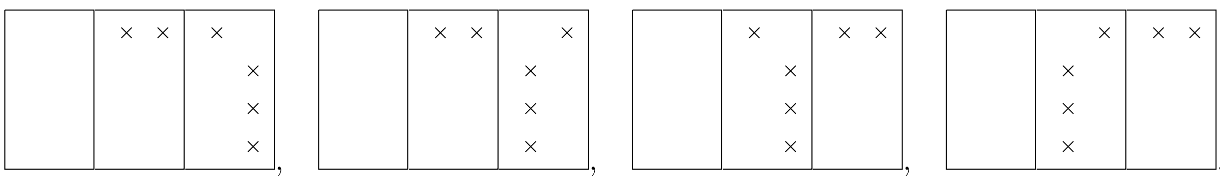
- X_1 represents 12, 18, 24, and 30.

- X_2 represents 1, 2, 22 and 23.

- X_3 represents 21.
- X_4 represents 7, 8, 13, 14, 28, 29, 34, and 35.
- X_5 represents Λ_1.
- X_6 represents 27 and 33.

Reassuringly, each MOG picture of the "Edited MOG" corresponds to exactly one of $X_1, \ldots X_6$. We constructed the Edited MOG by taking only those MOG pictures that form octads containing $\{14, \infty\}$, and so each choice of MOG picture (apart from Λ_1) gives four choices of hexad. If the brick tetrad contains $\{14, \infty\}$ then we have four choices for symbol in the square tetrad. If the square tetrad is the one that contains $\{14, \infty\}$ then we have two choices for the symbol in the brick tetrad and two choices for the position of the brick tetrad (in the middle or at the end of the square tetrad).

This is most easily seen in an example; MOG picture 12 represents four hexads of underlying structure X_1 given by



We also need to consider that each hexad represents several chambers because of the choice of duads. We need to split the six points into three 2-element sets. Suppose that the first two points we choose are the fixed duad, after choosing these two we are left with four points that need to be split into two duads. Choosing the first non fixed duad then defines the second, as it is just given by the remaining points. We recall that the order of the non fixed duads does not matter. This gives us $\binom{6}{2} \cdot \frac{\binom{4}{2}}{2} = 45$ chambers from each hexad.

Taking all these facts together we find that:

Remark 19.1.

$\lvert X_1 \rvert = 45 \cdot 4 \cdot 4 = 720$	$\lvert X_3 \rvert = 45 \cdot 4 \cdot 1 = 180$	$\lvert X_5 \rvert = 45 \cdot 1 \cdot 1 = 45$
$\lvert X_2 \rvert = 45 \cdot 4 \cdot 4 = 720$	$\lvert X_4 \rvert = 45 \cdot 4 \cdot 8 = 1440$	$\lvert X_6 \rvert = 45 \cdot 4 \cdot 2 = 360.$

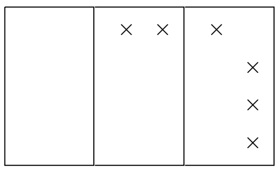
Where $|X_i|$ means the number of chambers with underlying type X_i.

For example,
$|X_1| =$ (# chambers from a hexad) $\times$ (# hexads from a MOG pic) $\times$ (# MOG pictures).
Summing across these 6 we find that the total number of chambers we have here is 3465. This coincides with our earlier calculation of the number of chambers, and so we know that none have been missed. We now know that two chambers with different underlying structures cannot lie in the same B-orbit, and every chamber must lie in a B-orbit with a representative from $X_1, \ldots, X_6$.

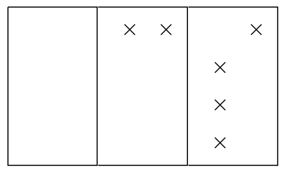
We claim that each of the underlying structures $X_1, \ldots, X_6$ is the representative of a single hexad B-orbit. We verify this in the case of X_1 and find the stabilizer of this hexad.

19.2 In the Case of X_1

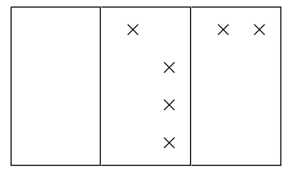
Consider hexads of type X_1. We noted that these arise from MOG pictures 12, 18, 24 and 30 and for each of these there were four hexads. We verify that these hexads, shown below, do form an orbit under the action of B.



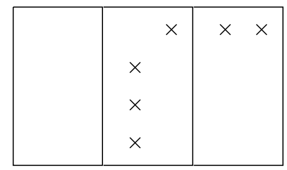
12.1



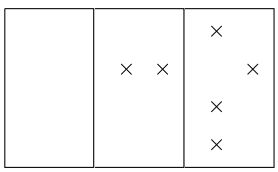
12.2



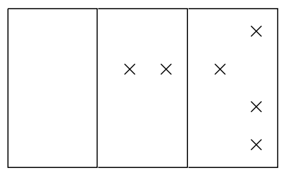
12.3



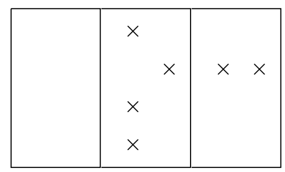
12.4



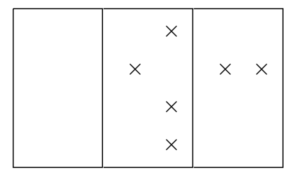
18.1



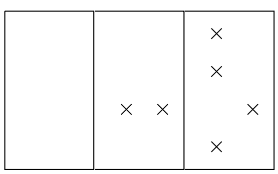
18.2



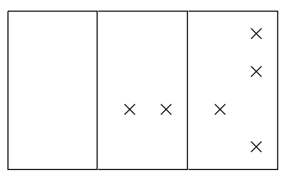
18.3



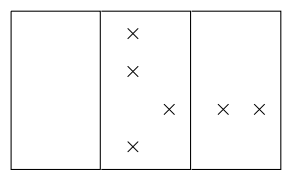
18.4



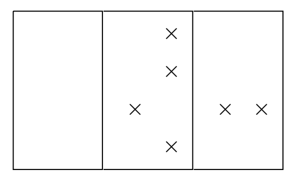
24.1



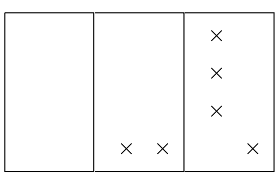
24.2



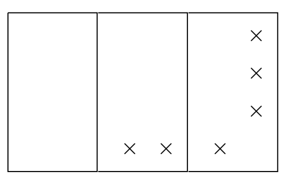
24.3



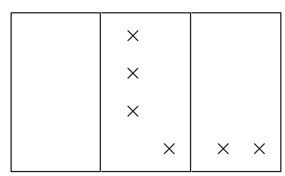
24.4



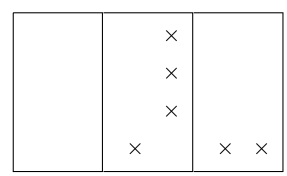
30.1



30.2



30.3



30.4

In order to show they are all in the same B-orbit let us verify that there is an element of B mapping the hexads labelled 12.1 to each of the others.

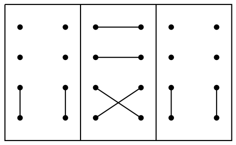
$12.1 \xrightarrow{b_4} 12.2$ $\quad 12.1 \xrightarrow{b_3} 12.3$ $\quad 12.1 \xrightarrow{b_3b_4} 12.4$

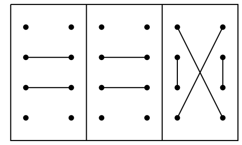
$12.1 \xrightarrow{b_5b_4} 18.1$ $\quad 12.1 \xrightarrow{b_5} 18.2$ $\quad 12.1 \xrightarrow{b_5b_3b_4} 18.3$ $\quad 12.1 \xrightarrow{b_5b_4} 18.4$

$12.1 \xrightarrow{b_6b_1} 24.1$ $\quad 12.1 \xrightarrow{b_6b_1b_4} 24.2$ $\quad 12.1 \xrightarrow{b_6b_1b_3} 24.3$ $\quad 12.1 \xrightarrow{b_6b_1b_3b_4} 24.4$

$12.1 \xrightarrow{b_6b_4} 30.1$ $\quad 12.1 \xrightarrow{b_6} 30.2$ $\quad 12.1 \xrightarrow{b_6b_3b_4} 30.3$ $\quad 12.1 \xrightarrow{b_5b_3} 30.4$

Hence the orbit of 12.1 has size at least sixteen. By the orbit stabilizer theorem, the stabilizer of 12.1 can be at most size eight. Therefore if we can show there are eight elements of B that stabilize 12.1 we can conclude that the orbit is of size 16 and is as given above.
The elements of B given by b_1b_3 and b_2, shown below, stabilize 12.1.

$b_1b_3 =$ 

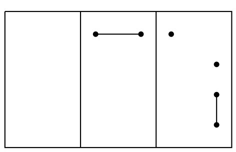
$b_2 =$ 

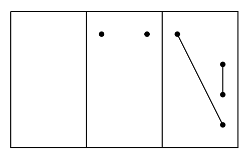
$(b_1b_3)b_2$ is of order 4 and so b_1b_3 and b_2 generate a subgroup of order at least eight, namely $Dih(8)$.
Resultantly the orbit of the hexad 12.1 is size 16 and is given by:
$\{$12.1, 12.2, 12.3, 12.4, 18.1, 18.2, 18.3, 18.4, 24.1, 24.2, 24.3, 24.4, 30.1, 30.2, 30.3, 30.4$\}$, using the labelling system above. And $Stab_B(X_1) = \langle b_1b_3, b_2 \rangle$.

We would now like to divide the chambers with underlying structure X_1 into chamber orbits. To do this we make use of the stabilizer of the hexad X_1 calculated above.
We take, x, a chamber with underlying structure X_1 and work out the size of the stabilizer of x as a subgroup of $Stab_B(X_1)$. This will be easier than working with the stabilizer of x in B as we will be working in a smaller group. Then we apply the orbit-stabilizer theorem to find the size of the $Stab_B(X_1)$-orbit in which x lies.

Remark 19.2. *This works because* $Stab_B(x) = Stab_{Stab_B(X_1)}(x)$.
If $g \in Stab_B(x)$ *if and only if* g *leaves* x *fixed, more generally it will also fix the underlying structure of* x. *Therefore* g *must lie in the subgroup of* B, $Stab_B(X_1)$, *and so* $g \in Stab_{Stab_B(X_1)}(x)$. *Now since* $Stab_B(X_1) \subseteq B$ *it follows that* $Stab_{Stab_B(X_1)}(x) \subseteq Stab_B(x)$. *And hence we have equality.*

It suffices to consider only the induced action of $Stab_B(X_1)$ on the points of X_1 because we are currently unconcerned with what happens to the other points of M_{22}.
The induced action on the points of X_1 is generated by the involutions below.

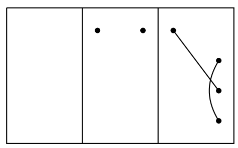
$b_1b_3 =$ 

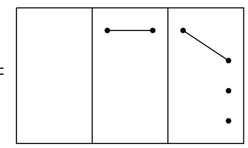
$b_2 =$ 

20 Chamber Orbits of M_{22}

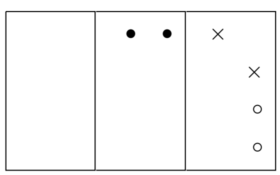
$\mathbf{X_1}$

It proves useful to work with the alternative set of generators of the induced action as given below.

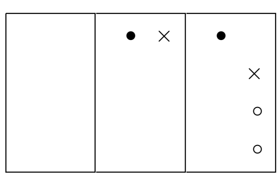
$x_1 =$ 

$x_2 =$ 

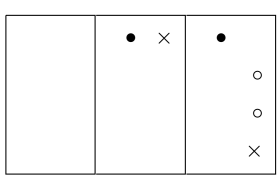
We now want to find a splitting of the hexads into duads, for each new choice we also want to check that they cannot be mapped to any previous choice by elements of B.
Again we use the notation $\bullet$ for points in the fixed duad, and $\circ$, $\times$ for those in the non fixed duads.

$X_{1,1} =$ 

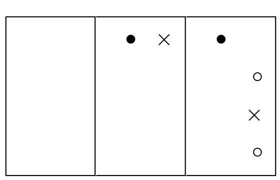
This is stabilized by x_1 and x_2 hence $|Stab_B(X_{1,1})| = |\langle x_1, x_2 \rangle| = 8$. By the orbit-stabilizer theorem, we find that the size of this orbit is 1. By Remark 19.1 we know this underlying structure represents $4 \cdot 4 = 16$ chambers. And so the size of the orbit with this representative element is of size 16.

$X_{1,2} =$ 

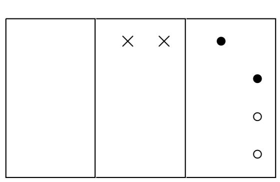
This is stabilized by only the identity and so the size of the orbit is 8. Using the same reasoning as before the size of the B-orbit represented by this element has size $8 \cdot 16 = 128$.

$X_{1,3} =$ 

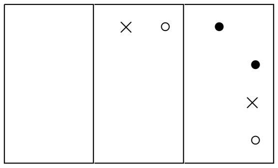
This is stabilized by only the identity and so the size of the orbit is 8. Therefore the B-orbit represented by this element has size 128.

$X_{1,4} =$ 

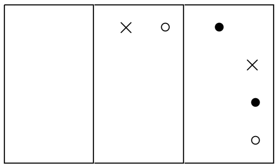
This is stabilized by only the identity and so the size of the orbit is 8. Therefore the B-orbit represented by this element has size 128.

$X_{1,5} =$ 

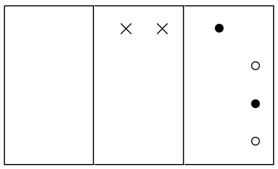
This is stabilized by x_2 and $(x_1x_2)^2$, giving us that the size of the stabilizer is 4. Therefore the B-orbit represented by this element has size $2 \cdot 16 = 32$.

$X_{1,6} =$ 

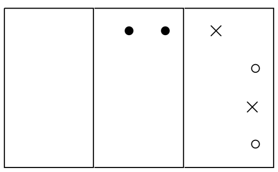
This is stabilized by $x_2(x_1x_2)^2$, giving us that the size of the stabilizer is 2. Therefore the B-orbit represented by this element has size $4 \cdot 16 = 64$.

$X_{1,7} =$ 

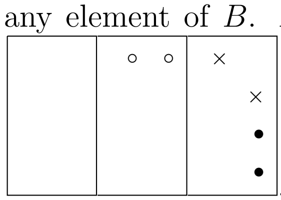
This is stabilized by only the identity and so the size of the orbit is 8. Therefore the B-orbit represented by this element has size 128.

$X_{1,8} =$ 

This is stabilized by x_1, giving us that the size of the stabilizer is 2. Therefore the B-orbit represented by this element has size 64.

$X_{1,9} =$ 

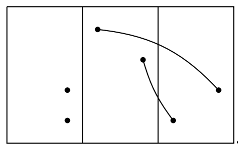
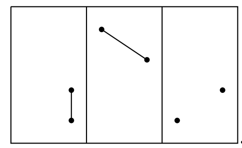
This is stabilized by x_1 and $(x_1x_2)^2$, giving us that the size of the stabilizer is 4. Therefore the B-orbit represented by this element has size 32.

We can check that we have all representatives by checking we have the right number of chambers. Here we have found $16 + 128 + 128 + 128 + 32 + 64 + 128 + 64 + 32 = 720$ as we have previously calculated.
Note that we picked these representatives rather deliberately. None of these representatives can be mapped to one another by any element of B. And so for example, once we have picked $X_{1,5}$ we then could not pick . This is because we can map from $X_{1,5}$ to this chamber by $x_1(x_1x_2)^2$, and so these are in the same B-orbit.

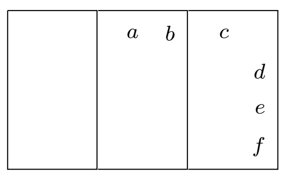
We repeat this process for each of the underlying structures. After finding the generators for the stabilizers for the other X_i we see that a lot of the cases are essentially the same as for X_1 just with a permutation of points in the underlying structure applied. The only different case is X_4.

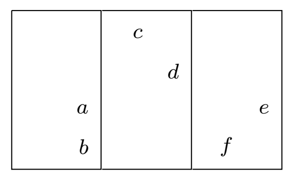
$\mathbf{X_2}$

To induced action of $Stab_B X_2$ on the points of X_2 is generated by:

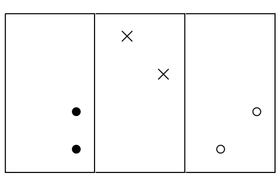
$x_1 =$ , $x_2 =$ .

There is a map from X_1 to X_2 given by $\begin{pmatrix} 17 & 11 & 22 & 5 & 9 & 6 \\ 20 & 18 & 17 & 5 & 13 & 21 \end{pmatrix}$. We are not currently interested in what happens to the other points of M_{22}, and so only need to consider where the point of the underlying structure are mapped. It is perhaps more clear when shown on an octad. The labelling of points is such that the letter a in X_1 is sent to the letter a in X_2 and so on.

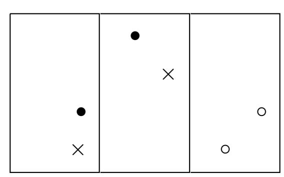
$X_1 =$ 

$X_2 =$ 

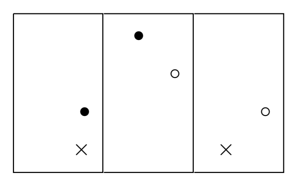
Using this map and making note of the fact that by Remark 19.1 each of these structures represents 16 chambers, we get the following orbits represented by chambers with the underlying structure of X_2.

$X_{2,1} =$ 

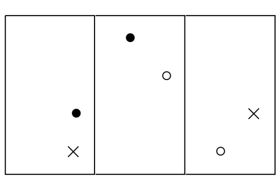
Orbit size = 16

$X_{2,2} =$ 

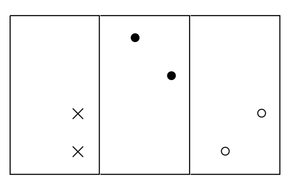
Orbit size = 128

$X_{2,3} =$ 

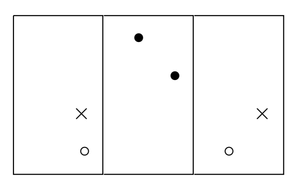
Orbit size = 128

$X_{2,4} =$ 

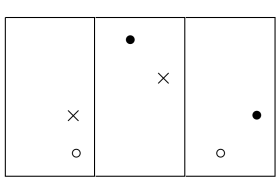
Orbit size = 128

$X_{2,5} =$ 

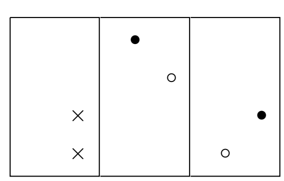
Orbit size = 32

$X_{2,6} =$ 

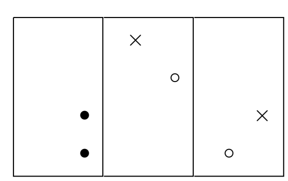
Orbit size = 64

$X_{2,7} =$ 

Orbit size = 128

$X_{2,8} =$ 

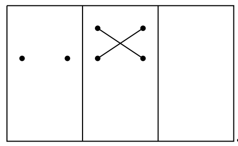
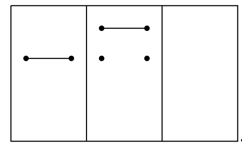
Orbit size = 64

$X_{2,9} =$ 

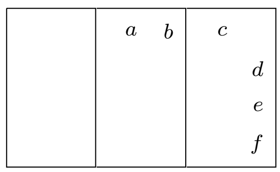
Orbit size = 32

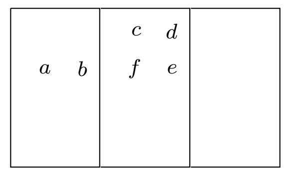
$\mathbf{X_3}$

To induced action of $Stab_B X_3$ on the points of X_3 is generated by:

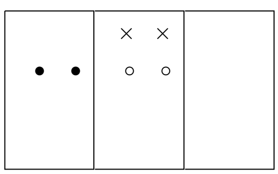
$x_1 =$  , $x_2 =$  .

There is a map from X_1 to X_3 given by $\begin{pmatrix} 17 & 11 & 22 & 5 & 9 & 6 \\ 0 & 8 & 17 & 13 & 11 & 4 \end{pmatrix}$.

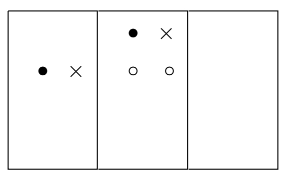
$X_1 =$ 

$X_3 =$ 

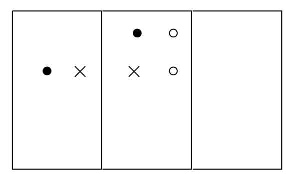
Using this map and making note of the fact that by Remark 19.1 each of these structures represents 4 chambers, we get the following orbits represented by chambers with the underlying structure of X_3.

$X_{3,1} =$ 

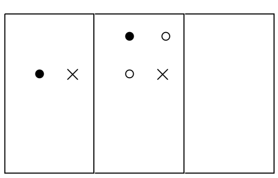
Orbit size = 4

$X_{3,2} =$ 

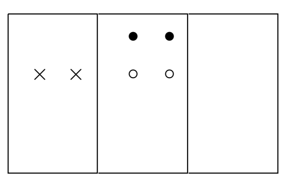
Orbit size = 32

$X_{3,3} =$ 

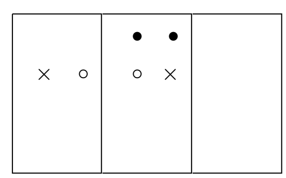
Orbit size = 32

$X_{3,4} =$ 

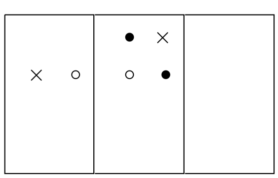
Orbit size = 32

$X_{3,5} =$ 

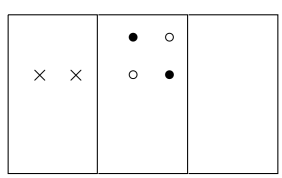
Orbit size = 8

$X_{3,6} =$ 

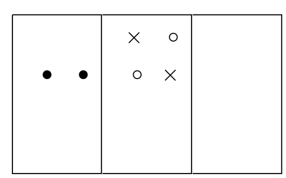
Orbit size = 16

$X_{3,7} =$ 

Orbit size = 32

$X_{3,8} =$ 

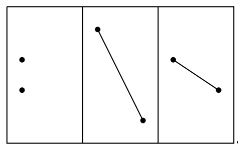
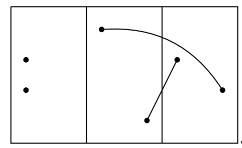
Orbit size = 16

$X_{3,9} =$ 

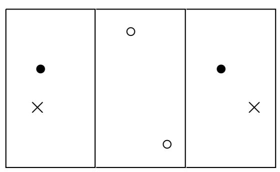
Orbit size = 8

$\mathbf{X_4}$

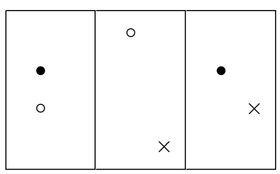
To induced action of $Stab_B X_4$ on the points of X_4 is generated by:

$x_1 =$ , $x_2 =$ .

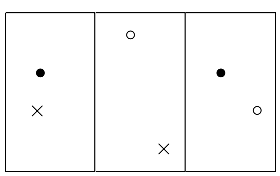
These generate a group order 4.

$X_{4,1} =$ 

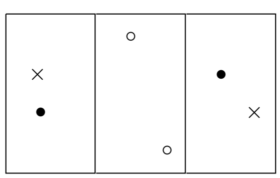
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so the size of the orbit is 4. By Remark 19.1 we know each of these represent 32 chambers we have that the size of the orbit with this representative is $4 \cdot 32 = 128$.

$X_{4,2} =$ 

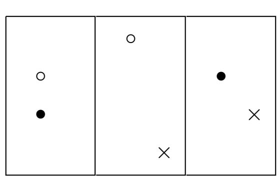
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,3} =$ 

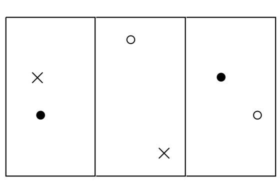
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,4} =$ 

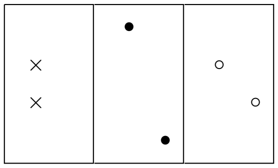
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,5} =$ 

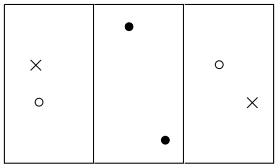
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,6} =$ 

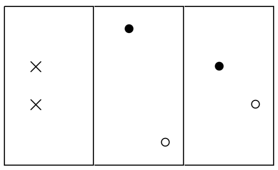
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,7} =$ 

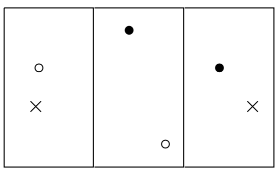
This is stabilized x_1, and so its stabilizer has size 2. The size of the orbit represented by this chamber is then $2 \cdot 32 = 64$.

$X_{4,8} =$ 

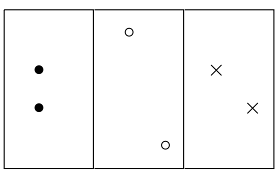
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,9} =$ 

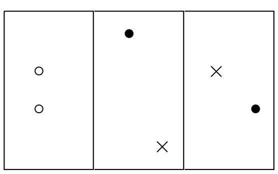
This is stabilized $x_1 x_2$, and so its stabilizer has size 2. The size of the orbit represented by this chamber is then 64.

$X_{4,10} =$ 

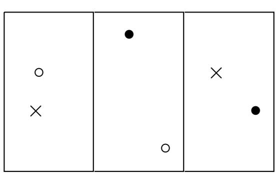
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,11} =$ 

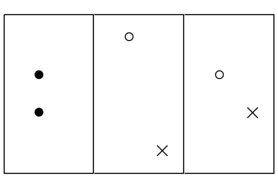
This is stabilized x_1, x_2, and so its stabilizer has size 4. The size of the orbit represented by this chamber is then $1 \cdot 32 = 32$.

$X_{4,12} =$ 

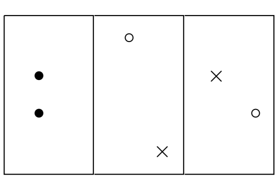
This is stabilized x_2, and so its stabilizer has size 2. The size of the orbit represented by this chamber is then 64.

$X_{4,13} =$ 

This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,14} =$ 

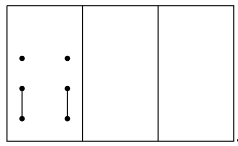
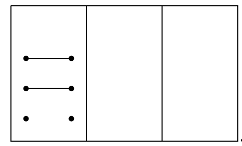
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

$X_{4,15} =$ 

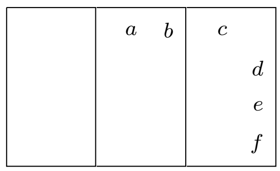
This is not stabilized by any of the non identity elements of $Stab_B X_4$ and so size of the orbit with this representative is 128.

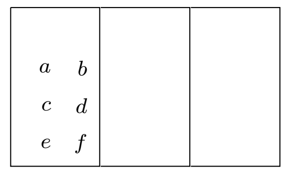
$\mathbf{X_5}$

To induced action of $Stab_B X_5$ on the points of X_5 is generated by:

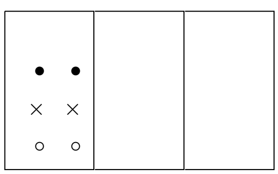
$x_1 =$ , $x_2 =$ .

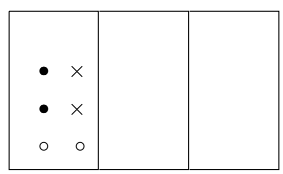
There is a map from X_1 to X_5 given by $\begin{pmatrix} 17 & 11 & 22 & 5 & 9 & 6 \\ 0 & 8 & 3 & 20 & 15 & 18 \end{pmatrix}$.

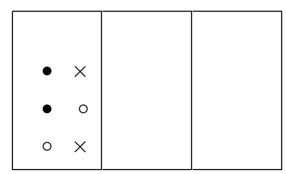
$X_1 =$ 

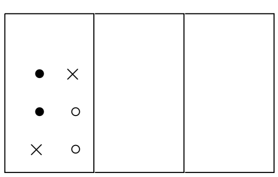
$X_5 =$ 

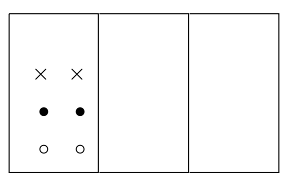
Using this map and making note of the fact that by Remark 19.1 each of these structures represents 1 chamber, we get the following orbits represented by chambers with the underlying structure of X_2.

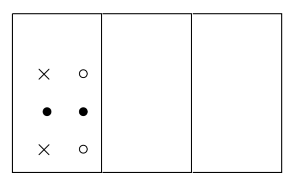
$X_{5,1} =$ 
Orbit size = 1

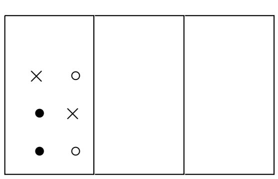
$X_{5,2} =$ 
Orbit size = 8

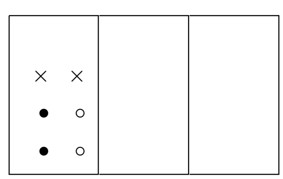
$X_{5,3} =$ 
Orbit size = 8

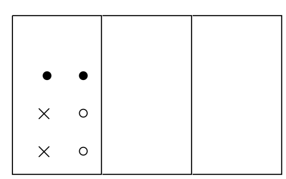
$X_{5,4} =$ 
Orbit size = 8

$X_{5,5} =$ 
Orbit size = 2

$X_{5,6} =$ 
Orbit size = 4

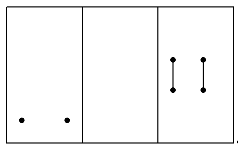
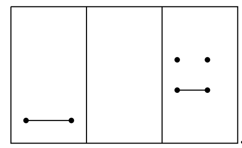
$X_{5,7} =$ 
Orbit size = 8

$X_{5,8} =$ 
Orbit size = 4

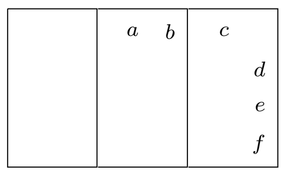
$X_{5,9} =$ 
Orbit size = 2

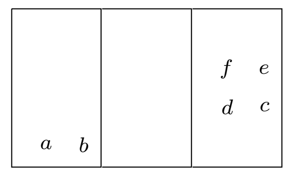
$\mathbf{X_6}$

To induced action of $Stab_B X_6$ on the points of X_6 is generated by:

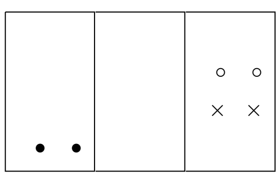
$x_1 =$ , $x_2 =$ .

There is a map from X_1 to X_6 given by $\begin{pmatrix} 17 & 11 & 22 & 5 & 9 & 6 \\ 15 & 18 & 5 & 12 & 9 & 1 \end{pmatrix}$.

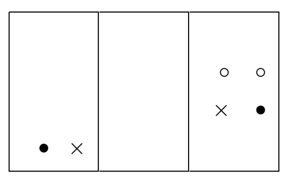
$X_1 =$ 

$X_6 =$ 

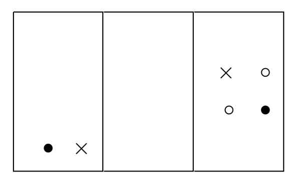
Using this map and making note of the fact that by Remark 19.1 each of these structures represents 8 chambers, we get the following orbits represented by chambers with the underlying structure of X_6.

$X_{6,1} =$ 

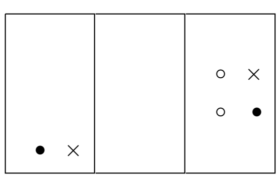
Orbit size = 8

$X_{6,2} =$ 

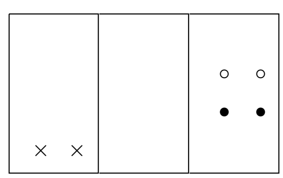
Orbit size = 64

$X_{6,3} =$ 

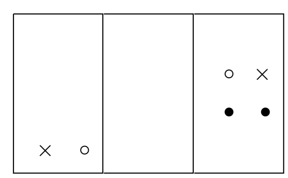
Orbit size = 64

$X_{6,4} =$ 

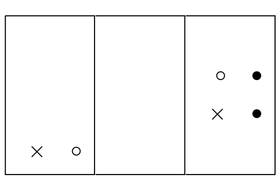
Orbit size = 64

$X_{6,5} =$ 

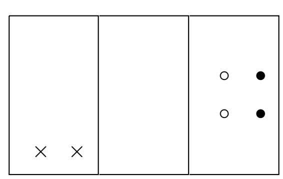
Orbit size = 16

$X_{6,6} =$ 

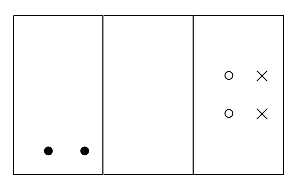
Orbit size = 32

$X_{6,7} =$ 

Orbit size = 64

$X_{6,8} =$ 

Orbit size = 32

$X_{6,9} =$ 

Orbit size = 16

21 The Jigsaw Game for M_{22}

We now play "The Jigsaw Game". To do this we make use of the fact that each B-orbit lies entirely in one disc by Theorem 5.12. And so to show that an entire B-orbit lies in a disc it suffices to show that only one element of the B-orbit lies in that disc.

Remark 21.1. *We know that there are 16 neighbours for each chamber. In order to calculate the chamber graph entirely by hand, we could calculate all neighbours of each representative as we did in Section 12. As we will see calculating neighbours for the M_{22} section is rather time consuming. So instead we use* MAGMA *only once to find the number of B-orbits in each disc.*
As a result once we have found the right number of B-orbits that lie in say disc i we can stop, and do not have to find any more neighbours of representatives in disc $i-1$. This means that when we calculate neighbours of chambers in disc i we know any new B-orbits we find must lie in disc $i+1$.
Without this information from MAGMA *we run the risk of missing some orbits in disc i because we did not calculate all neighbours of representatives in disc $i-1$. And so when we look at neighbours of representatives in disc i it could be the case that we are actually finding "sideways neighbours" that also lie in disc i.*

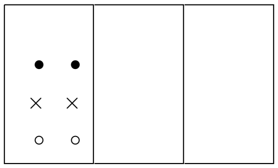
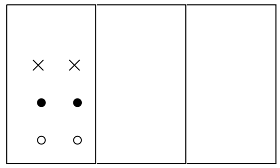
In section 12 calculating all neighbours enabled us to say much more than just the placement of B-orbits within the disc. We were able to draw the chamber graph fully. We then used this to calculate the sizes of the B-orbits and in turn the sizes of the discs. However the methods we have used for M_{22} means that we already have the sizes of the B-orbits. So although it is a shame to not have the computation done entirely by hand we have not lost much information. As discussed in the Remark above it would actually be possible for this to be done by hand by someone with more time and patience.

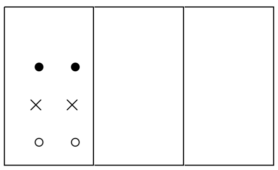
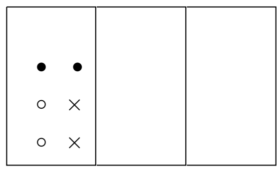
We are taking γ_0 to be $X_{5,1} = \begin{array}{|c|c|c|} \hline \bullet \ \bullet & \quad & \quad \\ \times \ \times & & \\ \circ \ \circ & & \\ \hline \end{array}$, which is the only element in disc 0 and is the only B-orbit of size 1.
To find the B-orbits in disc 1 we take neighbours of this chamber.
Although it is not necessary for disc 1, in future we will see that we get neighbours that do not explicitly appear in our list of representatives of B-orbits. As we know that each element must lie in a B-orbit, it remains to apply a sequence of elements of B to our neighbour until we reach a representative chamber. Similarly, sometimes it helps to first apply elements of B to our representative before we find a neighbour. In particular, this last trick proves helpful when we are nearing the end of The Jigsaw Game and we have only a few remaining pieces.

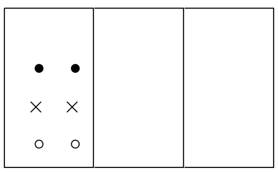
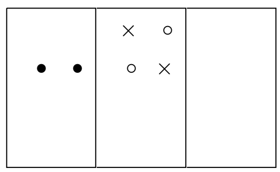
Throughout we use the notation $X_{i,j} \xrightarrow{T1} X_{k,l}$ meaning that the chambers $X_{i,j}$ and $X_{k,l}$ are type 1 neighbours. This means all chambers in the B-orbit represented by $X_{i,j}$ have type 1 neighbours in the B-orbit which contains $X_{k,l}$. For type 2 neighbours we use a slightly different notation. $X_{i,j} \xrightarrow[n]{T2} X_{k,l}$ means that the chambers $X_{i,j}$ and $X_{k,l}$ are type

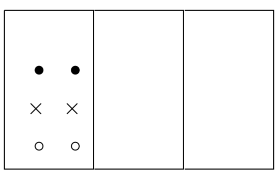
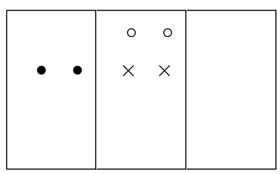
2 adjacent and that $X_{k,l}$ is a chamber arising from the MOG picture n. We also say that $X \xrightarrow{b_i} Y$ when the generating element b_i of B maps X to Y.

Disc 1

$X_{5,1} =$  $\xrightarrow{T1}$  $= X_{5,2}$

$X_{5,1} =$  $\xrightarrow[\Gamma_1]{T2}$  $= X_{5,3}$

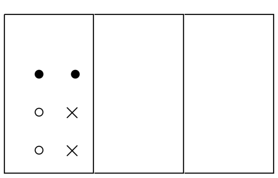
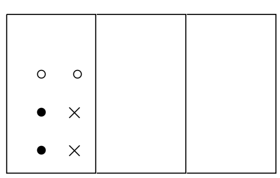
$X_{5,1} =$  $\xrightarrow[21]{T2}$  $= X_{3,9}$

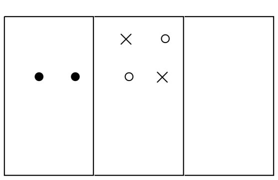
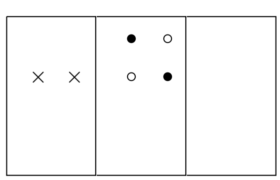
$X_{5,1} =$  $\xrightarrow[21]{T2}$  $= X_{3,1}$

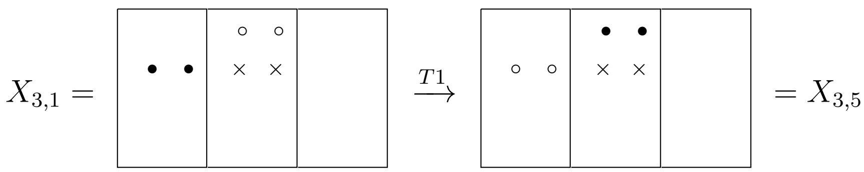
Using the fact that we know $\Delta_1(\gamma_0)$ has 4 orbits, we know we are done.
$\Delta_1(\gamma_0) = \{X_{5,2},\ X_{5,3},\ X_{3,9},\ X_{3,1}\}$ and the B-orbits of size 8, 4, 2, 2. So in total there are 16 chambers.

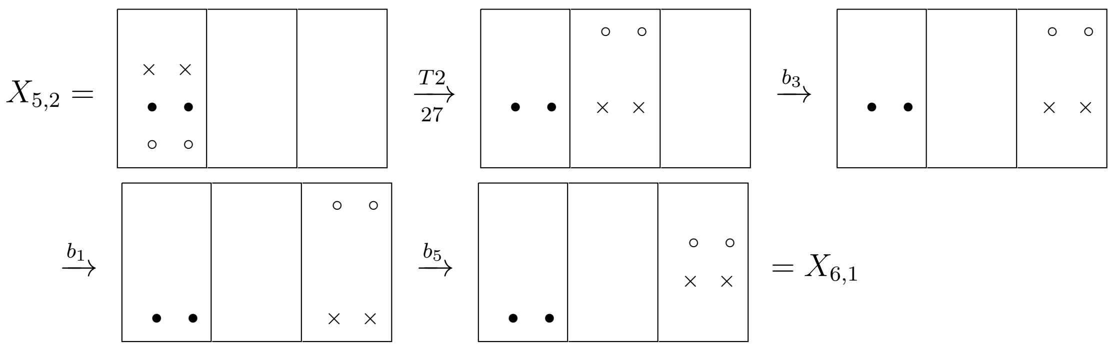
Because we have filled disc 1, we know any neighbours of a chamber in disc 1 we have not already seen must lie in disc 2.

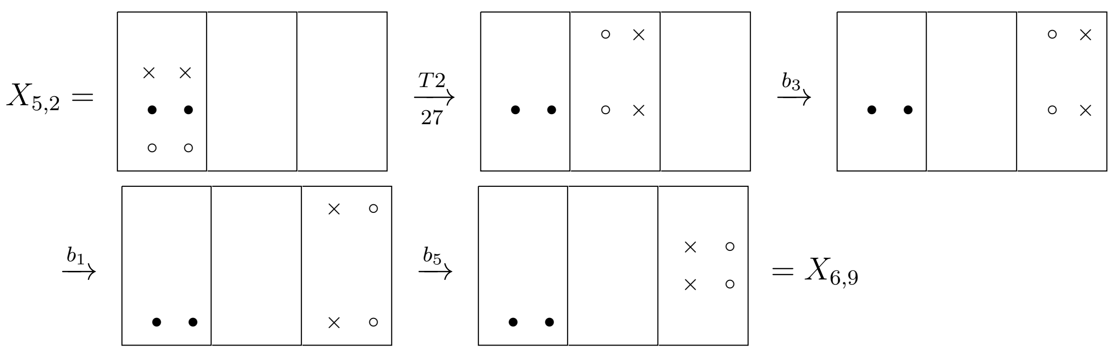
Disc 2

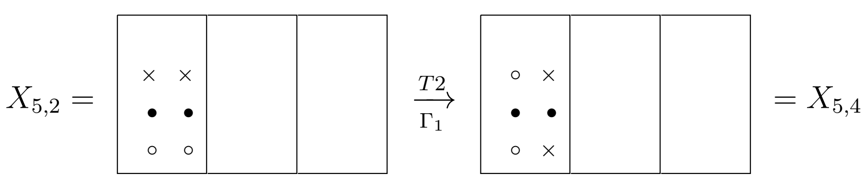
$X_{5,3} =$  $\xrightarrow{T1}$  $= X_{5,5}$

$X_{3,9} =$  $\xrightarrow{T1}$  $= X_{3,8}$



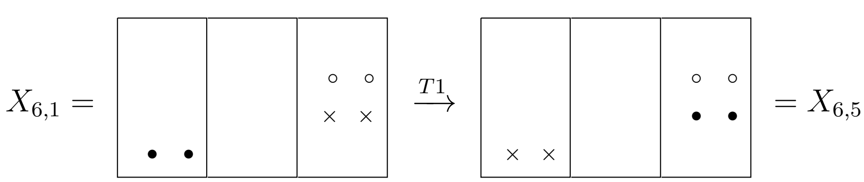




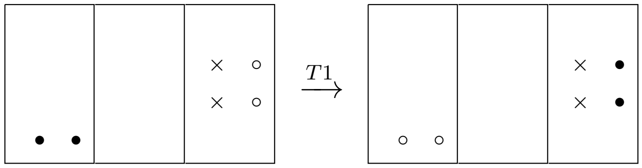


Again using the fact that we know that disc 2 has 6 B-orbits we know we are done! $\Delta_2(\gamma_0) = \{X_{5,5},\ X_{3,8},\ X_{3,5},\ X_{6,1},\ X_{6,9},\ X_{5,4}\}$ Then referring to our list of B-orbit representatives we know that the orbits of disc 2 have sizes 4, 4, 8, 8, 16, 16. And so in total we have 56 chambers in this disc.

Disc 3

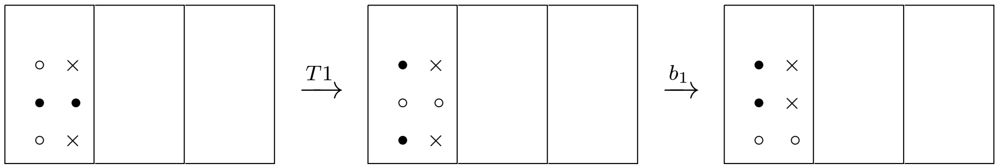


$X_{6,9} =$



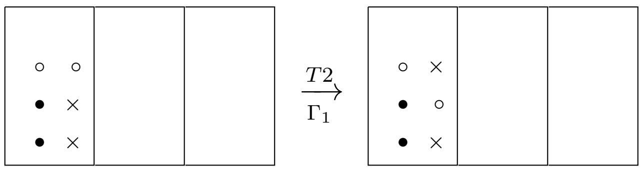
$= X_{6,8}$

$X_{5,4} =$



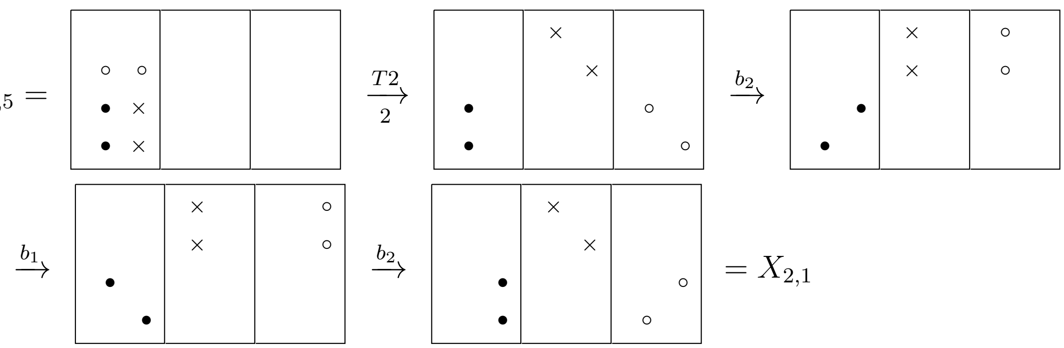
$= X_{5,7}$

$X_{5,5} =$



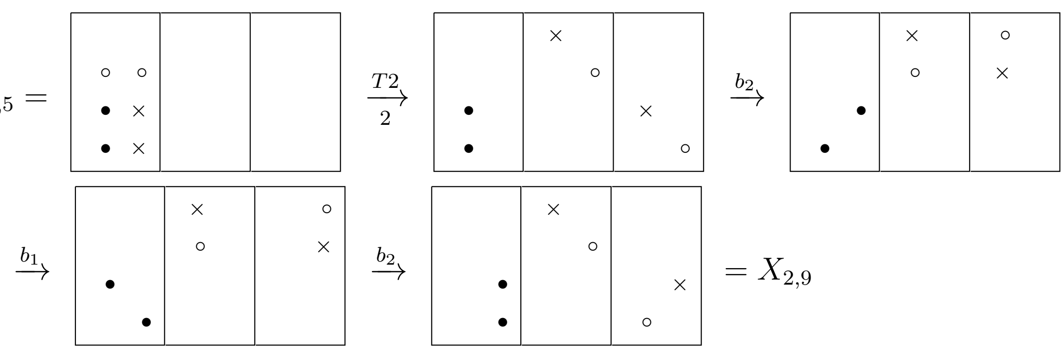
$= X_{5,6}$

$X_{5,5} =$



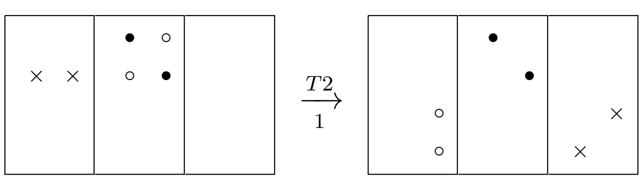
$= X_{2,1}$

$X_{5,5} =$

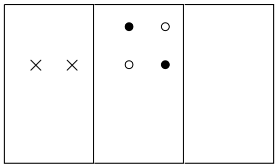
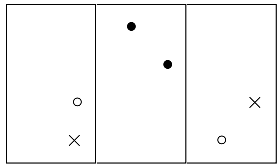
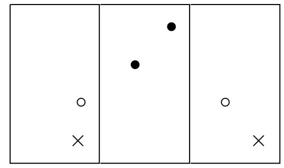


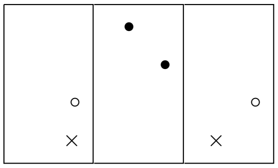
$= X_{2,9}$

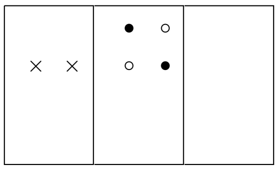
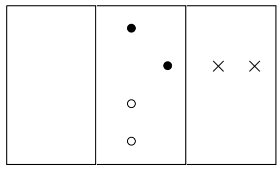
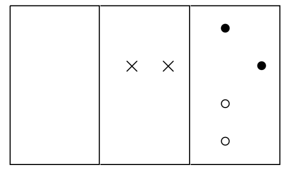
$X_{3,8} =$

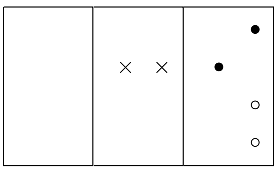
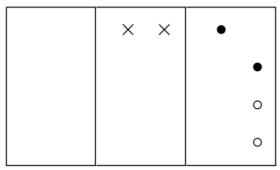


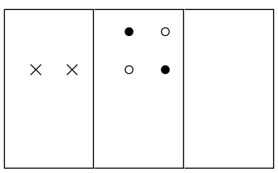
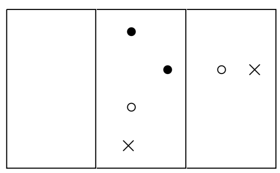
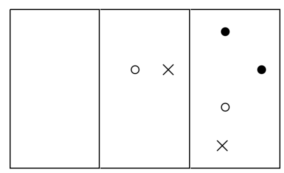
$= X_{2,5}$

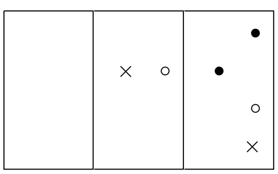
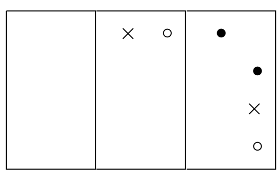
$X_{3,8} =$  $\xrightarrow[1]{T2}$  $\xrightarrow{b_5}$ 

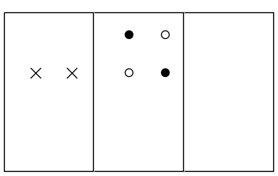
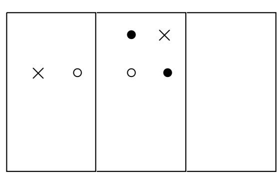
$\xrightarrow{b_4}$  $= X_{2,6}$

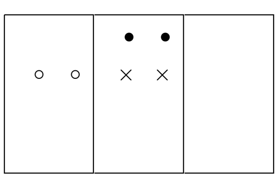
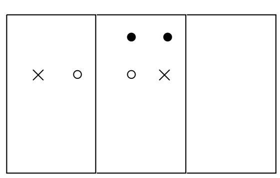
$X_{3,8} =$  $\xrightarrow[18]{T2}$  $\xrightarrow{b_3}$ 

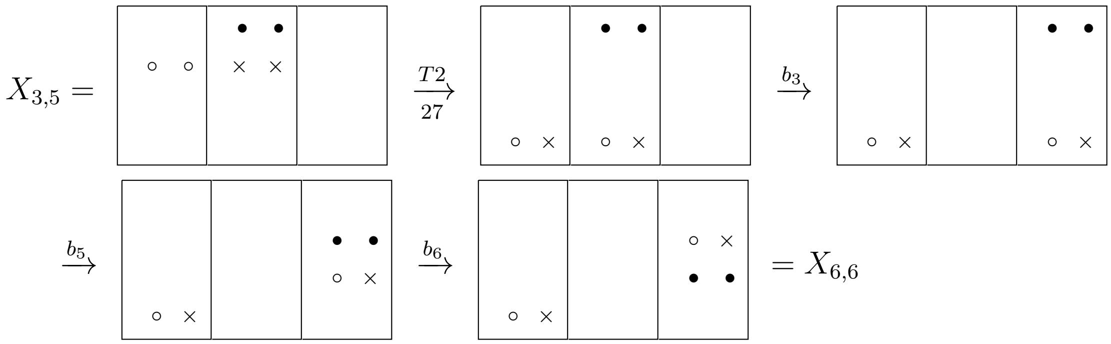
$\xrightarrow{b_4}$  $\xrightarrow{b_5}$  $= X_{1,5}$

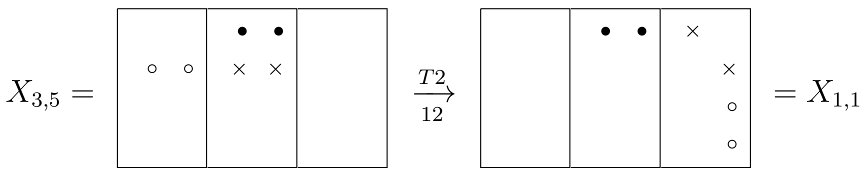
$X_{3,8} =$  $\xrightarrow[18]{T2}$  $\xrightarrow{b_3}$ 

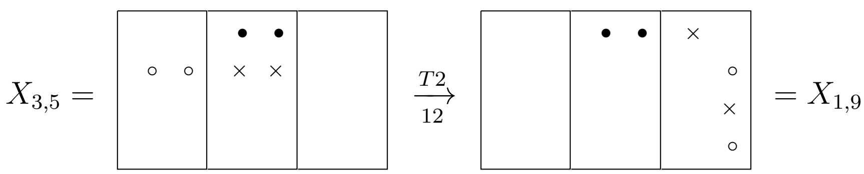
$\xrightarrow{b_4}$  $\xrightarrow{b_5}$  $= X_{1,6}$

$X_{3,8} =$  $\xrightarrow[21]{T2}$  $= X_{3,7}$

$X_{3,5} =$  $\xrightarrow[21]{T2}$  $= X_{3,6}$



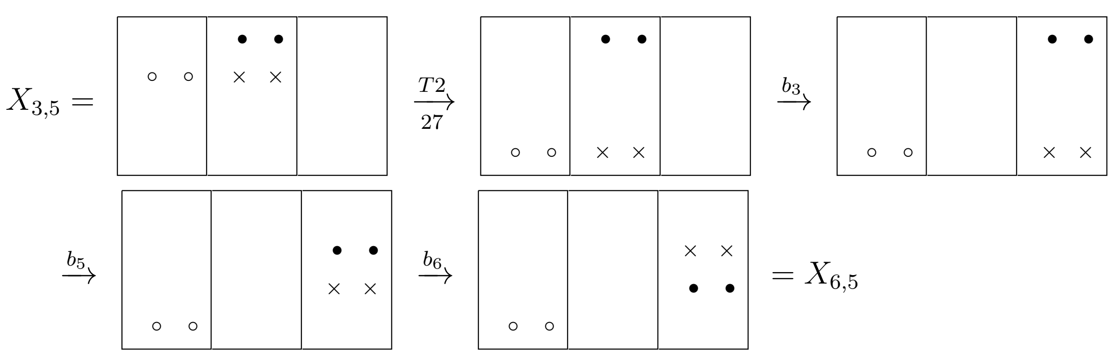




$\Delta_3(\gamma_0) = \{X_{6,5},\ X_{6,8},\ X_{5,7},\ X_{5,6},\ X_{2,1},\ X_{2,9},\ X_{2,5},\ X_{2,6},\ X_{1,5},\ X_{1,6},\ X_{3,7},\ X_{3,6},\ X_{6,6},\ X_{1,1},\ X_{1,9}\}$.

The B-orbits split 8, 8, 16, 16, 16, 16, 32, 32, 32, 32, 32, 32, 32, 64, 64.
So disc 3 has 15 B-orbits and 432 chambers.

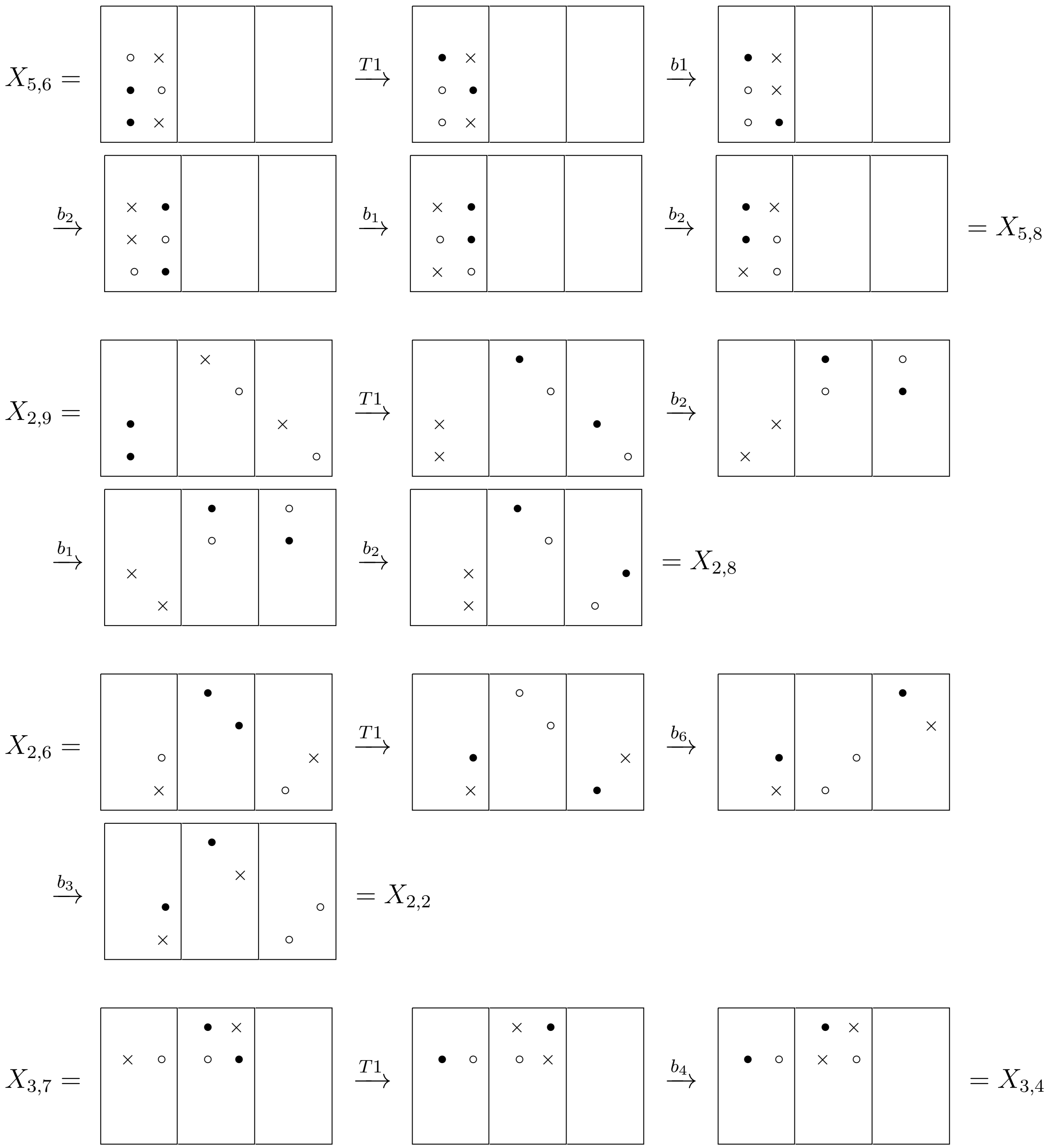
Although it might not appear so we have actually picked the choices of type 1 and type 2 neighbours rather deliberately. Some possible neighbours of chambers in disc 2 also lie in disc 2, so this method sometimes gives representatives we have already found. We see one of these explicitly below.

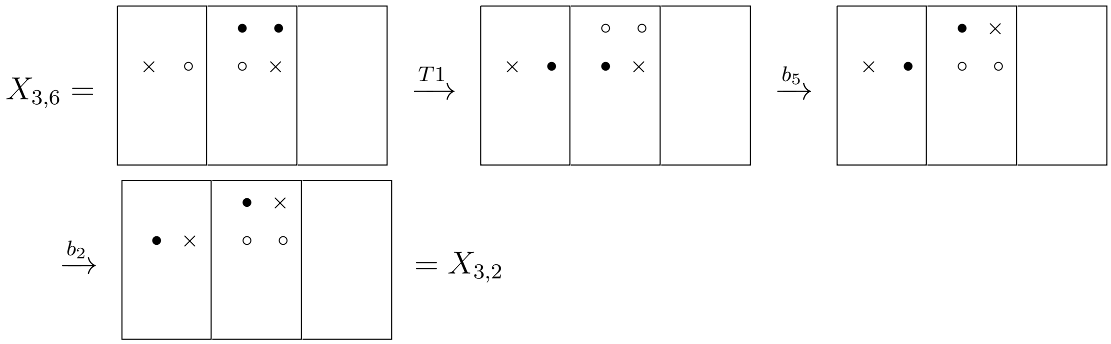


We know $X_{6,5}$ lies in $\Delta_2(\gamma_0)$, so we do not want to count this in the third disc. This type of thing can happen a lot in disc 4 because the chambers in disc 3 have a lot of "sideways neighbours" that are also in disc 3.
Calculating later discs is easier, this is where the jigsaw game really comes into play. We

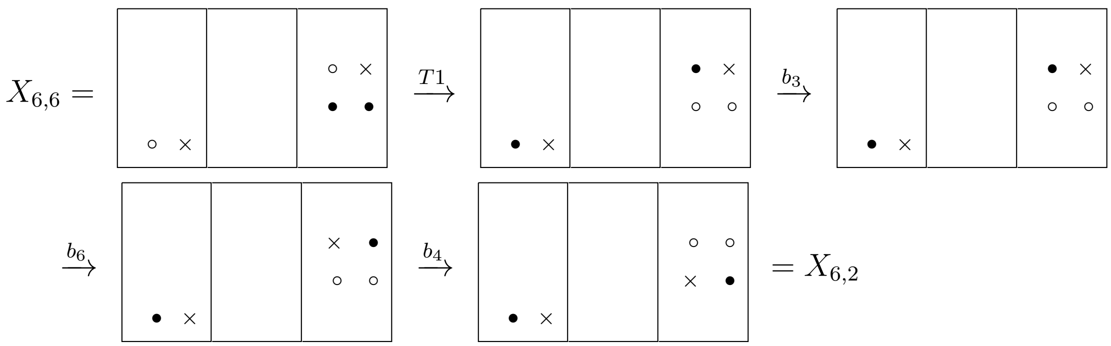
know which remaining B-orbits we are trying to fit in and so we can make strategic choices of type 1 and type 2 neighbours in order to reach them.

Disc 4

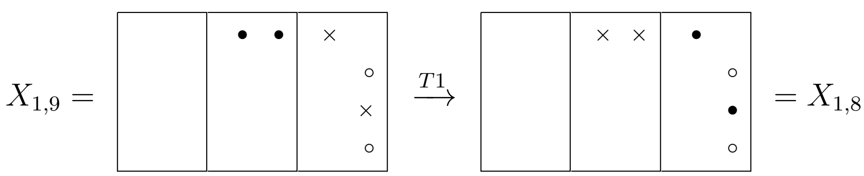




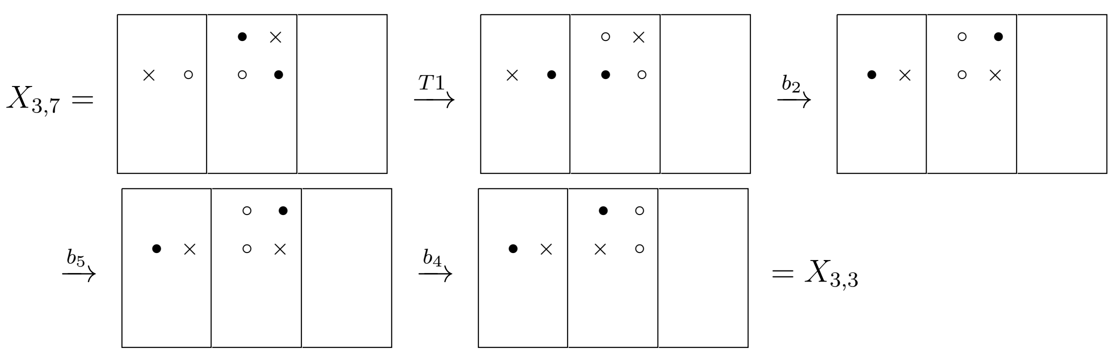
$X_{3,6} =$
$\xrightarrow{T1}$
$\xrightarrow{b_5}$
$\xrightarrow{b_2}$
$= X_{3,2}$



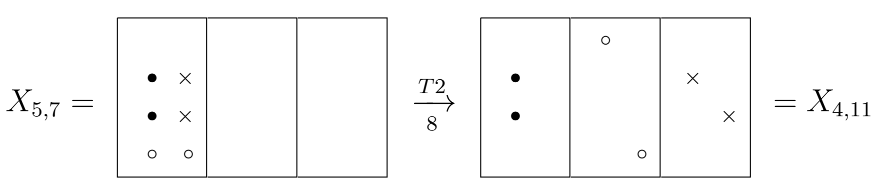
$X_{6,6} =$
$\xrightarrow{T1}$
$\xrightarrow{b_3}$
$\xrightarrow{b_6}$
$\xrightarrow{b_4}$
$= X_{6,2}$



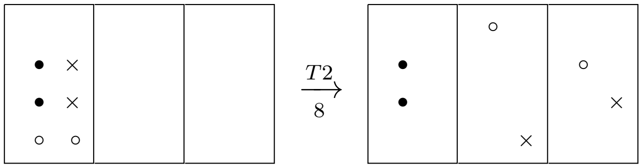
$X_{1,9} =$
$\xrightarrow{T1}$
$= X_{1,8}$



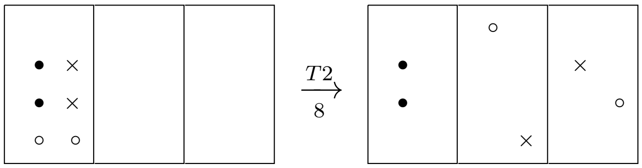
$X_{3,7} =$
$\xrightarrow{T1}$
$\xrightarrow{b_2}$
$\xrightarrow{b_5}$
$\xrightarrow{b_4}$
$= X_{3,3}$



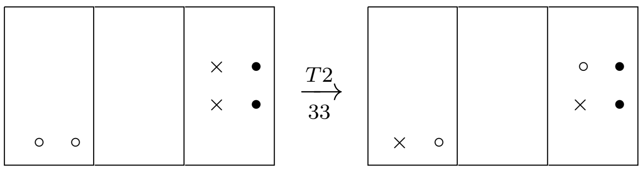
$X_{5,7} =$
$\xrightarrow[8]{T2}$
$= X_{4,11}$

$X_{5,7} =$ 

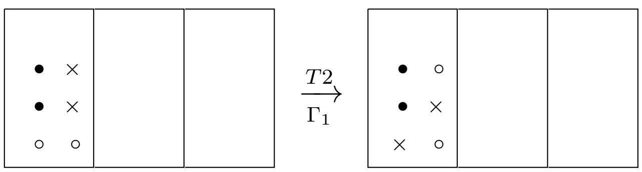
$= X_{4,14}$

$X_{5,7} =$ 

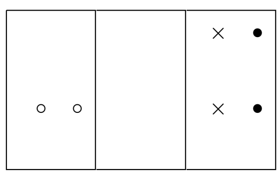
$= X_{4,15}$

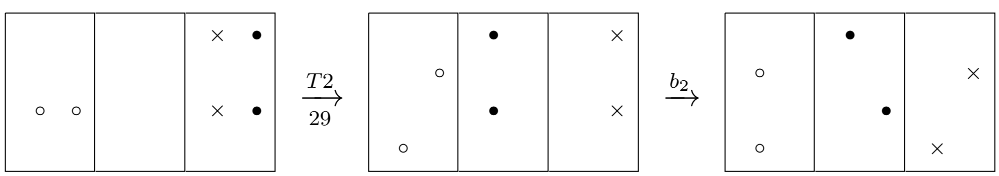
$X_{6,8} =$ 

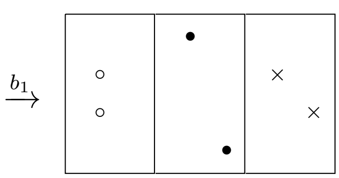
$= X_{6,7}$

$X_{5,7} =$ 

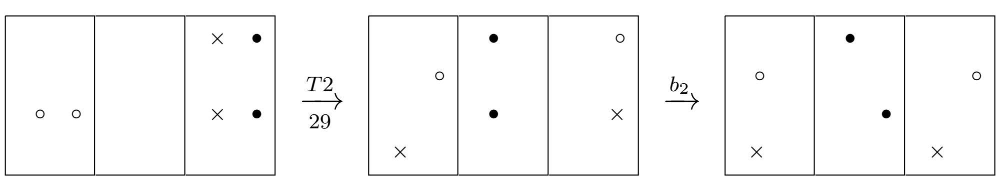
$= X_{5,9}$

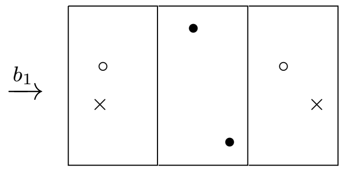
If we apply $b_1 b_5 b_3$ to $X_{6,8}$ we get  . This chamber will also lie in the B-orbit represented by $X_{6,8}$ and so we can use this in place of $X_{6,8}$ when calculating neighbours.

$b_1 b_5 b_3(X_{6,8}) =$ 



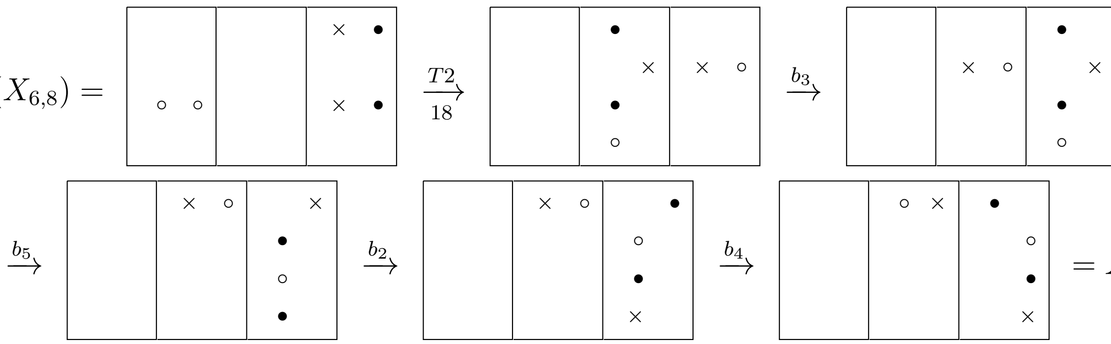
$= X_{4,7}$

$b_1 b_5 b_3(X_{6,8}) =$ 



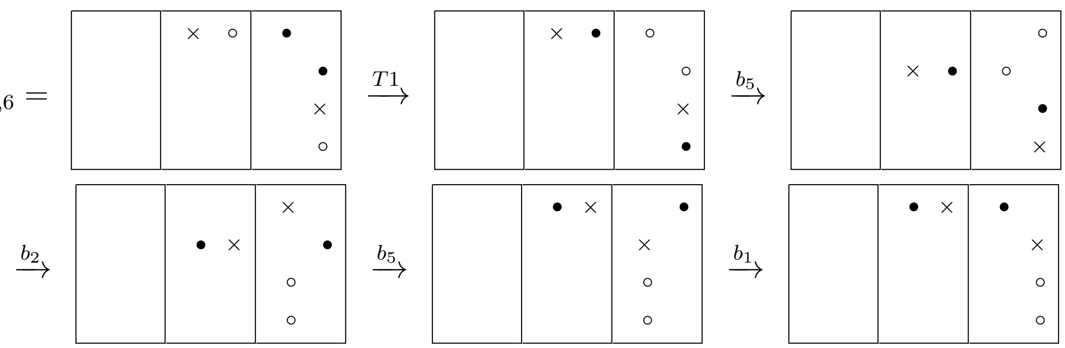
$= X_{4,8}$

$b_1 b_5 b_3(X_{6,8}) =$



$= X_{1,7}$

$X_{1,6} =$



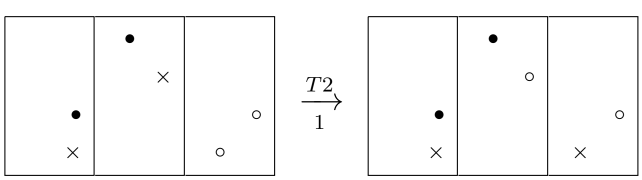
$= X_{1,2}$

$$\Delta_4(\gamma_0) = \left\{ \begin{array}{l} X_{5,8},\ X_{2,8},\ X_{2,2},\ X_{3,4},\ X_{3,2},\ X_{6,2},\ X_{1,8},\ X_{3,3},\ X_{4,11}, \\ X_{4,14},\ X_{4,15},\ X_{6,7},\ X_{5,9},\ X_{4,7},\ X_{4,8},\ X_{1,7},\ X_{1,2} \end{array} \right\}$$

And so disc 4 has 17 B-orbits, and by summing over the number of chambers in each B-orbit we find that disc 4 contains 1040 chambers.

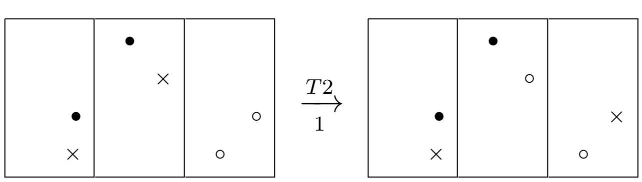
Disc 5

$X_{2,2} =$

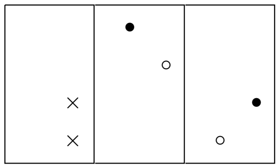
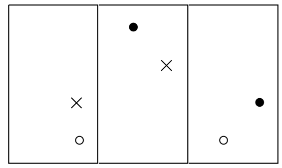


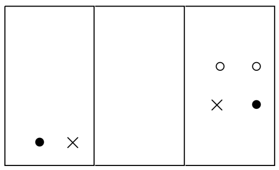
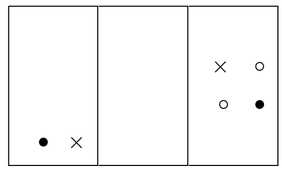
$= X_{2,3}$

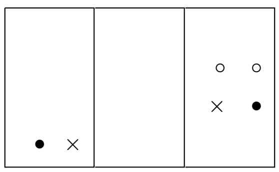
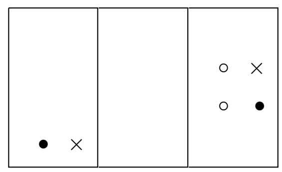
$X_{2,2} =$

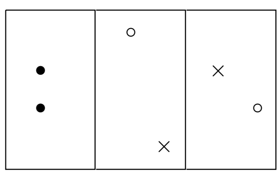
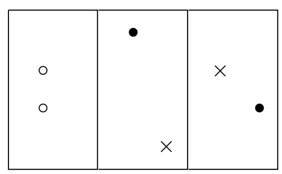


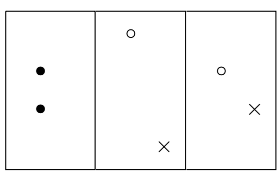
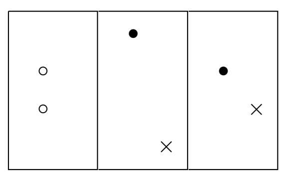
$= X_{2,4}$

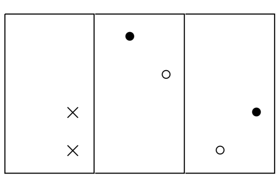
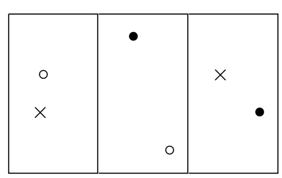
$X_{2,8} =$  $\xrightarrow[1]{T2}$  $= X_{2,7}$

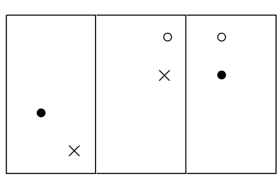
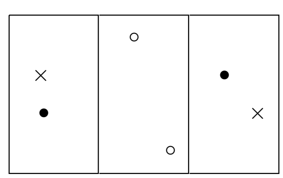
$X_{6,2} =$  $\xrightarrow[33]{T2}$  $= X_{6,3}$

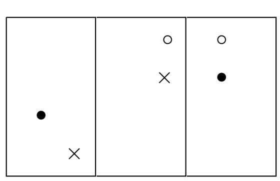
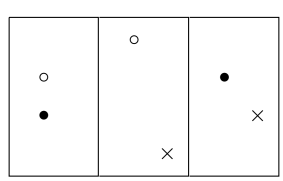
$X_{6,2} =$  $\xrightarrow[33]{T2}$  $= X_{6,4}$

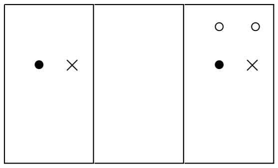
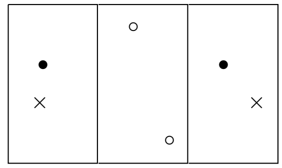
$X_{4,15} =$  $\xrightarrow{T1}$  $= X_{4,12}$

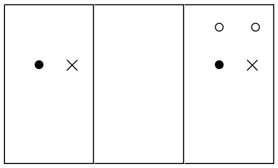
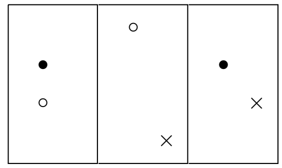
$X_{4,14} =$  $\xrightarrow{T1}$  $= X_{4,9}$

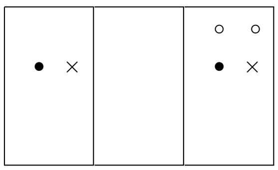
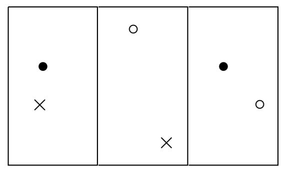
$X_{2,8} =$  $\xrightarrow[8]{T2}$  $= X_{4,13}$

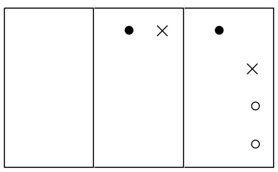
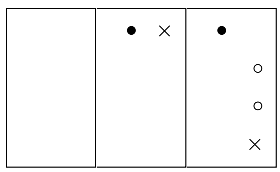
$b_5 b_3 b_2(X_{2,3}) =$  $\xrightarrow[8]{T2}$  $= X_{4,4}$

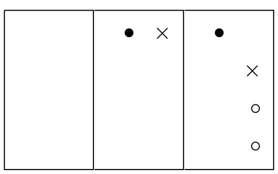
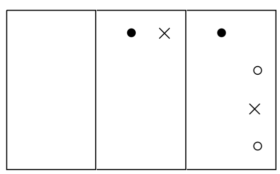
$b_5 b_3 b_2(X_{2,3}) =$  $\xrightarrow[8]{T2}$  $= X_{4,5}$

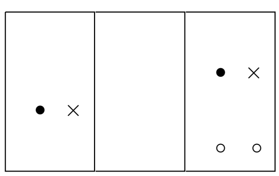
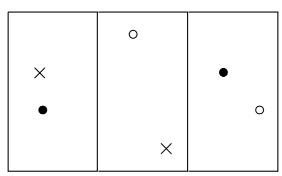
$b_5b_3(X_{3,2}) =$  $\xrightarrow[8]{T2}$  $= X_{4,1}$

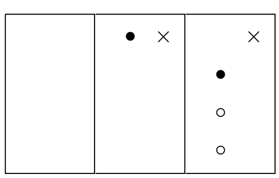
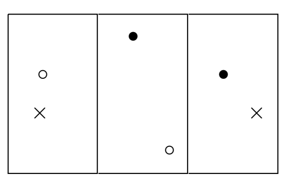
$b_5b_3(X_{3,2}) =$  $\xrightarrow[8]{T2}$  $= X_{4,2}$

$b_5b_3(X_{3,2}) =$  $\xrightarrow[8]{T2}$  $= X_{4,3}$

$X_{1,2} =$  $\xrightarrow[12]{T2}$  $= X_{1,3}$

$X_{1,2} =$  $\xrightarrow[12]{T2}$  $= X_{1,4}$

$b_5b_6b_1(X_{6,2}) =$  $\xrightarrow[8]{T2}$  $= X_{4,6}$

$b_2b_1b_2(X_{1,2}) =$  $\xrightarrow[8]{T2}$  $= X_{4,10}$

$$\Delta_5(\gamma_0) = \left\{ \begin{array}{l} X_{2,3},\ X_{2,4},\ X_{2,7},\ X_{6,3},\ X_{6,4},\ X_{4,12},\ X_{4,9},\ X_{4,13},\ X_{4,4}, \\ X_{4,5},\ X_{4,1},\ X_{4,2},\ X_{4,3},\ X_{1,3},\ X_{1,4},\ X_{4,6},\ X_{4,10} \end{array} \right\}$$

Hence disc 5 has 17 B-orbits, summing over the number of chambers in each B-orbit we find that disc 5 contains 1920 chambers.
We have used all the B-orbits up, and so we know that the diameter of the chamber graph

is 5.
Finally we have found the following disc structure.

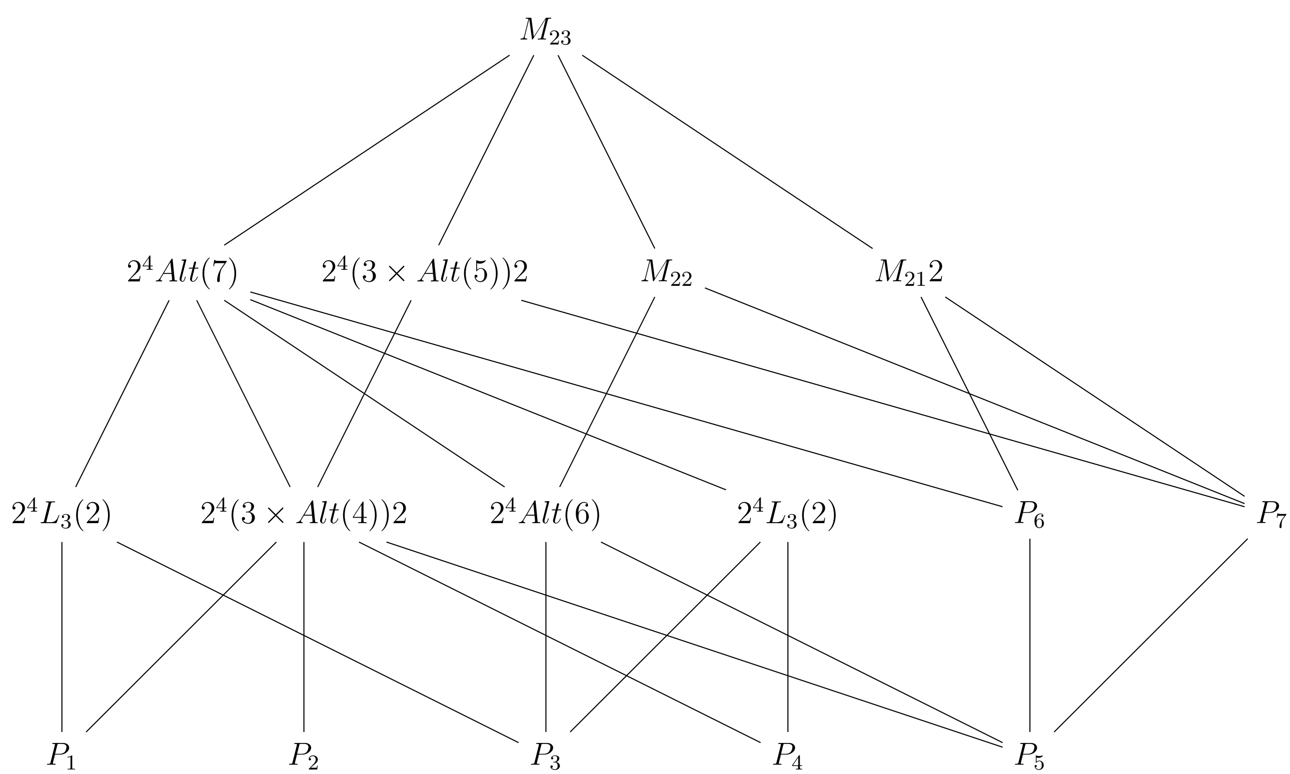
i^{th} Disc	1	2	3	4	5
$\lvert\Delta_i(\gamma_0)\rvert$	16	56	432	1040	1920
Number of B-orbits	4	6	15	17	17

This took much longer than using Magma but it gives us a more combinatorial view of what is going on.

22 M_{23} using Magma

Unlike the other Mathieu groups M_{23} has two non isomorphic geometries, and so we calculate the disc structure of the chamber graphs of both of them in this section.

The Hasse diagram for M_{23} that appears in [24] is shown below.



We can see from the diagram that all P_is apart from P_7 are within $2^4Alt(7)$. This means that without P_7 we cannot generate anything outside of $2^4Alt(7)$, and in particular it is not possible to generate M_{23}. So any minimal parabolic system must contain P_7.
Recall that we are looking for geometric systems, and so we require $P_J \cap P_K = P_{J\cap K}$ where

$P_J = \langle P_j | j \in J \subseteq I \rangle$ and $P_\emptyset = B$.
We know that any system must contain P_7 and so cannot contain P_6. This is because if it did we should have $P_{\{6\}} \cap P_{\{7\}} = P_\emptyset = B$, but we can see from the diagram that it actually contains the bigger group P_5.
All remaining $P_i's$ apart from P_3 are contained in $2^4(3 \times Alt(5))2$. And so by the same reasoning as for P_7 we know that the system must contain P_3.
Both P_7 and P_3 are trapped in M_{22} therefore we need to add more $P_i's$ in order to generate the whole of M_{23}.
Can the system contain P_2? No because $P_{\{2,3\}} \cap P_{\{2,7\}}$ contains $2^4(3 \times Alt(4))2$ which is bigger than P_2. And so if we add P_2 then the system will not be geometric.
However we get no problems if we add P_1 or P_4. $\{P_1, P_3, P_7\}$ and $\{P_4, P_3, P_7\}$ both generate M_{23}, consequently there are the two geometric systems.
These two systems are non isomorphic as there is no automorphism of M_{23} that interchanges P_1 and P_4. These correspond to different choices of $L_3(2)$ conjugacy class in $Alt(7)$.
In [26], the first geometry is referred to as the "1-geometry" and the second the "3-geometry". This references the fact that the copy of $2^4L_3(2)$ corresponding to the 1-geometry is the stabilizer of a 1 dimensional subspace of 2^4 when we treat 2^4 as a vector space over $\mathcal{F}_2$, and the copy of $2^4L_3(2)$ corresponding to the 3-geometry is the stabilizer of a 3 dimensional space of 2^4.

Although we will not form the chamber graph of M_{23} by hand it is interesting to understand what the objects of each type are.
M_{23} forms a rank 3 system and so there are 3 types of objects.
Type 1 objects are given by points of Ω, and so there are 23 type 1 objects.
Type 2 objects are triads, sets of three points. And so there are $\binom{23}{3} = 1771$ of these.
Type 3 objects are a little more complicated. At first we choose any heptad. In the MOG this corresponds to forming M_{23} by fixing a single point of M_{24}, heptads are then given by all octads of the MOG containing this fixed point. This means there are $\frac{\binom{23}{4}}{\binom{7}{4}} = 253$ heptads.
We then need to form a Fano plane with these 7 points. Note there are 30 choices of Fano plane for each 7 points.
The 30 Fano planes form a single conjugacy class under the action of $Sym(7)$. The splitting criterion [28], tells us that either this conjugacy class must be preserved under the action of $Alt(7)$ or splits into two conjugacy classes of equal size. We know that these Fano planes must divide between the two systems, and so in particular divide equally between the two systems. Resultantly each geometry has 15 Fano planes.
These are divided evenly between the two systems, and so in each geometry of M_{23} there 15 choices of Fano plane. Therefore there are $253 \times 15 = 3795$ choices of type 3 objects.

A chamber is given by an object of each type, with type 1 incident with type 2 and type 2 incident with type 3.
A type 1 object, a point, is incident with a type 2, a trio, if the point is one of the elements of the trio. A type 2 object, a trio, is incident with a type 3, a Fano plane formed from a heptad, if the trio is one of the lines of the Fano plane.

How many chambers are there?
We have 3795 choices of type 3 object. There are 7 choices of trio within a type 3 object, corresponding to the 7 lines of the Fano plane. Then there are 3 choices of a point lying within any trio. Consequently there are $3795 \times 7 \times 3 = 79695$ chambers in each system.

How many neighbours of each type does a chamber have?
For type 1 neighbours we need to fix the type 2 and type 3 objects. For fixed type 2 and type 3 object we then get three choices for the type 1 object. One for each of the points in the trio of the type 2 object. And so any chamber has two type 1 neighbours.

For type 2 neighbours we fix the type 1 and type 3 objects. This means we need to find how many lines there are on a Fano plane containing a particular point. It is clear to see from the diagram in Section 6.1 that the answer is three. And so there are three chambers all with the same type 1 and type 3 objects. And so each chamber has two type 2 neighbours.

For type 3 neighbours we need to find how many chambers contain the same type 1 and type 3 neighbours. And so we need to find how many Fano planes we can pick containing a given line.
First, we calculate how many heptads there are containing these three points, say $\{x_1, x_2, x_3\}$. This is the same as finding how many octads contain a given four points, $\{x_1, x_2, x_3, x_4\}$, where we have added in the point $x_4 \in \Omega$ is such that $M_{23} = Stab_{M_{24}}(x_4)$.
Therefore there are $\frac{\binom{20}{1}}{\binom{4}{1}} = 5$ choices of heptad containing these three points.
Now we have the heptads we need to work out how many Fano planes contain the type 2 object as a line.
All lines of the Fano plane are the same, and so we can place the 3 points forming a line anywhere we would like. Wherever we choose to place the fourth point it will always be on a common line with each of the three points, and so we can place this where we choose. We are then left with three remaining points to place, for each of the $3! = 6$ possible combinations we get a different Fano plane. Ans so there are 6 Fano planes containing a given line.
Hence there are $6 \times 5 = 30$ Fano planes that contain the type 2 and type 1 object. However again we are dividing these between the two systems and so in each geometry, there are 15 containing this trio. As a result each chamber has 14 type 3 neighbours.

This gives us that the total valency is 18.

In MAGMA when forming M_{23} we took the same copy of M_{24} as we did when forming M_{22}. This time rather than fixing both ∞ and 14 we just fix ∞. This means that the conjugate of M_{22} that we have previously used is now a subgroup of M_{23}.
M_{22} and M_{23} have the same power of two in the prime factorisation of their orders, thus we can conclude they share the same Sylow 2 subgroups. When forming $B = Syl_2(M_{23})$ we can simply take the same generators found in Section 18.

The diameter of the chamber graph of the first geometry of M_{23}, the "1-geometry", is 7. The disc structure is as below.

i^{th} Disc	1	2	3	4	5	6	7
$\lvert\Delta_i(\gamma_0)\rvert$	18	92	664	3104	10728	36032	29056
Number of B-orbits	5	13	32	81	157	318	228

The diameter of the chamber graph of the second geometry of M_{23}, the "3-geometry", is 7. The disc structure is as below.

i^{th} Disc	1	2	3	4	5	6	7
$\lvert\Delta_i(\gamma_0)\rvert$	18	92	664	3104	10728	36544	28544
Number of B-orbits	5	13	32	81	157	322	224

It is interesting to note that both geometries have the same disc structure up to and including the fifth disc.

In each case we can see that we do get valency 18 and 79695 chambers.

23 M_{24} Summary

M_{24} has already been extensively studied in [3], so here I simply summarise some of the findings for completeness.

The types of objects of M_{24} are given below.

1. The type 1 object are the sextets of the MOG, as defined in Section 16,
2. Type 2 objects are fours groups (which are groups that are isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$), say $F = \{id, y_1, y_2, y_3\}$ such that $\{Fix_\Omega(y_1), Fix_\Omega(y_2), Fix_\Omega(y_3)\}$ is a trio.
3. Type 3 objects are involutions of G that fix an octad point-wise.

The incidence of objects is given below.

1. A type 1 object, S, is incident with a type 2 object, $F = \{1, y_1, y_2, y_3\}$, if y_i stabilises S for $i = 1 : 3$, and the trio $\{Fix_\Omega(y_1), Fix_\Omega(y_2), Fix_\Omega(y_3)\}$ can be obtained by pairing of tetrads of S.
2. A type 2 object, F, is incident with a type 3 object, y, if y is one of the elements of F.
3. A type 1 object, S, is incident with a type 3 object, y, if y stabilises S and $Fix_\Omega(y)$ is an octad that can be obtained by taking the union of two tetrads of S.

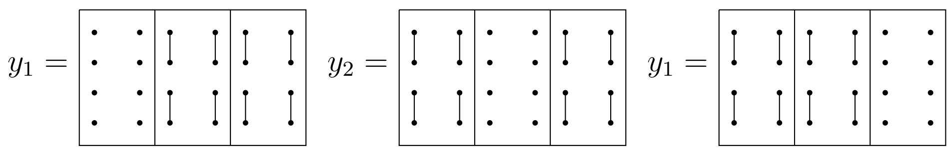
For example in [3], they choose γ_0 to be the chamber with the following objects.

1. The sextet is given by $S =$

1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6
1	2	3	4	5	6

.

2. $F = \{1, y_1, y_2, y_3\}$ where:



3. The type 3 object is given by $y = y_1$.

Let us check the incidence of each object.
Clearly each y_i stabilises S, and in addition $Fix_\Omega(y_1)$ is the octad given by taking the union of the tetrads labelled 1 and 2 in S, $Fix_\Omega(y_2)$ is the octad given by taking the union of the tetrads labelled 3 and 4 in S, and $Fix_\Omega(y_3)$ is the octad given by taking the union of the tetrads labelled 5 and 6 in S. So this type 1 object is incident with this type 2 object.
The type 2 object is incident with the type 3 because $y = y_1$.
The type 1 object is incident with the type 3 because as already discussed $y = y_1$ stabilises S, and $Fix_\Omega(y) = Fix_\Omega(y_1)$ is the octad given by taking the union of the tetrads labelled 1 and 2 in S.
And so taking all these objects together gives us a single chamber of the chamber graph of M_{24}.

There is a concise way to represent this chamber.

1∗ 2∗	3∗ 4∗	5∗ 6∗
1∗ 2∗	3∗ 4∗	5∗ 6∗
1 2	3 4	5 6
1 2	3 4	5 6

$$\underline{1,2} \mid 3,4 \mid 5,6$$

The numbering 1,. . . ,6 shows the tetrads in the sextet that forms our type 1 object. The asterisks show which pairs of points within a tetrad are interchanged, a point x is interchanged with point y if and only if they are in the same tetrad and are either both marked with an asterisk or neither marked with an asterisk. So this gives our type 2 object. The partition of tetrads $1, 2 \mid 3, 4 \mid 5, 6$ shows our trio, here we get the trio $\{\Lambda_1, \Lambda_2, \Lambda_3\}$. And finally the underlining of $\{1, 2\}$ shows that we take out type 3 object to be the involution that fixes the union of these two tetrads.

The diameter of the chamber graph of M_{24} is 17, and the disc structure is as below.

i^{th} Disc	1	2	3	4	5	6	7	8	9	10	11
$\lvert\Delta_i(\gamma_0)\rvert$	6	20	56	144	368	848	1800	3810	8040	16920	32832
Number of B-orbits	3	5	7	9	13	18	24	31	39	53	71

12	13	14	15	16	17
55200	62336	47616	6656	2048	384
93	78	47	10	6	2

.

As with M_{12} the last disc of M_{24} contains few B-orbits, and so it was studied in more depth.

We choose a chamber from each of the two B-orbits in $\Delta_{17}(\gamma_0)$, denote these $\Delta^1_{17}(\gamma_0)$ and $\Delta^2_{17}(\gamma_0)$. $|\Delta^1_{17}(\gamma_0)| = 128$ and $|\Delta^2_{17}(\gamma_0)| = 256$. Each chamber in $\Delta^1_{17}(\gamma_0)$ is 1-adjacent to two chambers in $\Delta^2_{17}(\gamma_0)$.

The geodesic closure of a representative chamber in both of these orbits has been calculated and shown below.

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\lvert\overline{\{\gamma_0, \Delta^1_{17}(\gamma_0)\}} \cap \Delta_i(\gamma_0)\rvert$	1	4	8	24	24	16	16	16	24	24	16	16	16	24	24	8	4	1

and

Disc i of $\mathcal{C}(\Gamma)$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\lvert\overline{\{\gamma_0, \Delta^2_{17}(\gamma_0)\}} \cap \Delta_i(\gamma_0)\rvert$	1	4	12	28	36	44	42	50	50	50	50	42	44	36	28	12	4	1

.

Hence $|\overline{\{\gamma_0, \Delta^1_{17}(\gamma_0)\}}| = 266$ and $|\overline{\{\gamma_0, \Delta^2_{17}(\gamma_0)\}}| = 534$.

The maximal opposite sets of M_{24} are size 3, and these form 14 orbits under the action of M_{24}.

24 Code

Below is the "standard code" that forms the chamber graph. Before we input this we have to generate G, B and each P_i within MAGMA for each group. The standard code then computes the chamber graph of the given system as described in Example 5.8 and makes use of Remark 5.10 in order to identify cosets with B-orbits.
Specifically, this code is for a system of rank 3 with each P_i having index 2 in B. For structures with different rank or valency the code requires minor editing in places such as `NeighboursofB`, `Neighbours[1]` and `BorbitsDiscs[1]`.

24.1 Standard Code

```
DB:=DoubleCosetRepresentatives(G,B,B);
We use the elements of DB to represent B-orbits.
TrP1:=Transversal(P1,B);
TrP2:=Transversal(P2,B);
TrP3:=Transversal(P3,B);
These commands give an indexed sets of right coset reps for B in P1,P2,P3.
NeighboursofB:=[TrP1[2],TrP1[3],TrP2[2],TrP2[3],TrP3[2],TrP3[3]];
TrPi[1] is the identity so corresponds to B, hence we do not use it as a neighbour
TrPi[j], for j=2,3 are both i adjacent.
for i:=1 to #DB do
if DB[i] in P1 then a:=i;end if;end for;
for i:=1 to #DB do
if DB[i] in P2 then b:=i;end if;end for;
for i:=1 to #DB do
 if DB[i] in P3 then c:=i;end if;end for;
We're finding the B-orbits that each neighbour of B lies in.

Neighbours:=[ ];Neighbours[1]:=[a,a,b,b,c,c];
This gives which B-orbits are adjacent with the fixed chamber, listed in order of
adjacency type from 1 to 3.
BorbitsDiscs:=[];BorbitsDiscs[1]:={a,b,c};
So we now know these B orbits lie in the first disc.

Done:={1} join BorbitsDiscs[1];
We are collecting all the B-orbits that we have already placed in a disc.
Left:={x: x in [1..#DB]};
while #Done ne #DB do
While not all B-orbits have been placed in a disc we need to continue with the code.
l:=#BorbitsDiscs;l;

sum:=0;for c in BorbitsDiscs[l] do sum:=sum + Index(B,B meet B^DB[c]);end for;sum;
Here we calculate the size of each B orbit and sum across those in the lth
disc. We then print the value.
```

```
#BorbitsDiscs[l];
```
Printing the number of B orbits in the lth disc.
```
for j in BorbitsDiscs[l] do temp:=[ ];
for i:=1 to #NeighboursofB do temp:=Append(temp,NeighboursofB[i]*DB[j]);
```
We're forming a sequence of all chambers linked to the chambers in Disc l.
```
end for;

temp1:=[ ];
for k:=1 to #temp do g:=temp[k];
Trg:=Transversal(B,B meet B^g);
flag:=false;
for m in Left do h:=DB[m];
```
For each representative in Left (the orbits yet to be placed in a disc), form the DB element it represents.
```
for n:=1 to #Trg do
if g*Trg[n]*h^-1 in B then temp1[k]:=m;flag:=true;break;end if;
end for;
```
Is the B-orbit represented by h (h represented by m) adjacent to the B-orbit represented by g. We are doing this because we want each coset represented by a single element so we have to account for the possibility of two different elements representing the same coset.
```
if flag eq true then break;end if;
```
The command flag means that we can break the loop once we find which coset we are in. Each element lies in only one B-orbit so once we have one there's no point searching through the remaining. This saves computational time.
```
end for;
end for;
Neighbours[j]:=temp1;
end for;

temp2:={ };
```
This is going to be the set that contains all B orbits which are connected to the B-orbits in the lth disc. This will include neighbours within the disc l as well as backward and forward neighbours.
```
for j in BorbitsDiscs[l] do
temp2:= temp2 join{x: x in Neighbours[j]};
end for;
if l eq 1 then BorbitsDiscs[l+1]:=temp2 diff (BorbitsDiscs[l] join 1);end if;
```
We have the B-orbits in disc 1 and 2, so to know the number of B-orbits in the second disc we need only throw away the first disc and B.
```
if l ne 1 then BorbitsDiscs[l+1]:=temp2 diff (BorbitsDiscs[l] join BorbitsDiscs[l-1]);
end if;
```
If we are in a different disc then we have calculated forward and backward neighbours, so to find the number of B-orbits in the lth disc we need to throw

```
away the ones we've gathered from the disc before and after.
Done:=Done join BorbitsDiscs[l+1];
if l eq 1 then Left:=Left diff {1};end if;
If we are in first disc then what is left is everything apart from the
fixed chamber.
if l ne 1 then Left:=Left diff BorbitsDiscs[l-1];end if; If we arre in the lth disc
then what is left is everything that was left previously with the B-orbits in the
(l-1)th disc removed.
end while;

This  next bit of code is to deal with the last disc.
d:=#BorbitsDiscs;
Left:=BorbitsDiscs[d] join  BorbitsDiscs[(d-1)];
B-orbits in the last disc can only be connected with those in the same disc or the
previous one.
for j in BorbitsDiscs[d] do temp:=[ ];
for i:=1 to #NeighboursofB do temp:=Append(temp,NeighboursofB[i]*DB[j]);
end for;

temp1:=[ ];
for k:=1 to #temp do g:=temp[k];
Trg:=Transversal(B,B meet B^g);
flag:=false;
for m in Left do h:=DB[m];
for n:=1 to #Trg do
if g*Trg[n]*h^-1 in B then temp1[k]:=m;flag:=true;break;end if;
end for;
if flag eq true then break;end if;
end for;
end for;
Neighbours[j]:=temp1;
end for;
```

24.2 Code for $GL_4(2)$

```
K<w>:=FiniteField(2);
GL42:=GeneralLinearGroup(4,K);
y:=GL42![0,0,0,1, 0,0,1,0, 0,1,0,0, 1,0,0,0];
ans1:={}; for x in GL42 do t:=y[1]*x; if t eq y[1] then ans1:=ans1 join {x}; end if;
end for;
PP1:=sub<GL42|ans1>;
ans2:={}; for x in GL42 do t:=y[1]*x; if t in {y[1], y[2], y[1]+y[2]} then ans2:=ans2
 join {x}; end if; end for;
ans3:={}; for x in GL42 do t:=y[2]*x; if t in {y[1], y[2], y[1]+y[2]} then ans3:=ans3
join {x}; end if; end for;
T:=ans2 meet ans3;
PP2:=sub<GL42|T>;
ans4:={}; for x in GL42 do t:=y[1]*x; if t in {y[1], y[2], y[3],
y[1]+y[2], y[1]+y[3], y[2]+y[3], y[1]+y[2]+y[3]} then ans4:=ans4 join {x}; end if;
end for;
ans5:={}; for x in GL42 do t:=y[2]*x; if t in {y[1], y[2], y[3],
y[1]+y[2], y[1]+y[3], y[2]+y[3], y[1]+y[2]+y[3]} then ans5:=ans5 join {x}; end if;
end for;
ans6:={}; for x in GL42 do t:=y[3]*x; if t in {y[1], y[2], y[3],
y[1]+y[2], y[1]+y[3], y[2]+y[3], y[1]+y[2]+y[3]} then ans6:=ans6 join {x}; end if;
end for;
Q:=ans4 meet ans5 meet ans6;
PP3:=sub<GL42|Q>;
G:=GL42;
P1:=PP1 meet PP2;
P2:=PP1 meet PP3;
P3:=PP2 meet PP3;
B:=P1 meet P2 meet P3;
f,g:=IsIsomorphic(G, Alt(8));
ans:={}; for x in B do ans:=ans join {g(x)}; end for;
BB:=sub<Alt(8)|ans>;
ans:={}; for x in P1 do ans:=ans join {g(x)}; end for;
PP1:=sub<Alt(8)|ans>;
ans:={}; for x in P2 do ans:=ans join {g(x)}; end for;
PP2:=sub<Alt(8)|ans>;
ans:={}; for x in P3 do ans:=ans join {g(x)}; end for;
PP3:=sub<Alt(8)|ans>;

DB:=DoubleCosetRepresentatives(G,B,B);

TrP1:=Transversal(P1,B);
TrP2:=Transversal(P2,B);
TrP3:=Transversal(P3,B);
```

```
NeighboursofB:=[ TrP1[2],TrP1[3], TrP2[2],TrP2[3], TrP3[2],TrP3[3]];
for i:=1 to #DB do
if DB[i] in P1 then a:=i;end if;end for;
for i:=1 to #DB do
if DB[i] in P2 then b:=i;end if;end for;
for i:=1 to #DB do
 if DB[i] in P3 then c:=i;end if;end for;

Neighbours:=[];Neighbours[1]:=[a,a,b,b,c,c];
BorbitsDiscs:=[];BorbitsDiscs[1]:={a,b,c};
```

24.3 Code for M_{22}

```
S24:=Sym(24);
a1:=S24!(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23);
b1:=S24!(16,8,15,6,11,21,18,12,23,22,20)(4,7,13,2,3,5,9,17,10,19,14);
c1:=S24!(24,1)(16,4)(8,14)(15,19)(6,10)(11,17)(21,9)(18,5)(12,3)(23,2)(22,13)(20,7);
d1:=S24!(1,4,16)(9,21,19)(5,17,11)(14,8,3)(2,13,22)(10,6,7);
G:=sub<S24|a1,b1,c1,d1>;F:=Stabilizer(G,{24,1,4,16});
GG:=Stabilizer(G,1) meet Stabilizer(G,2);
B:=Sylow(GG,2);
NGGB:=NormalSubgroups(B);
#NGGB;
for i:=1 to #NGGB do if #NGGB[i]`subgroup eq 16 and
 IsElementaryAbelian(NGGB[i]`subgroup) then i; end if; end for;
Q:=Normalizer(GG, NGGB[8]`subgroup);
P:=Normalizer(GG, NGGB[10]`subgroup);
TT:=Transversal(Q,B);
K:=P meet Q;
for i:=1 to #TT do H:=sub<GG| B, TT[i]>; if #H eq 3*2^7 and sub<GG|H,K> eq Q then
P1:=H; end if; end for;
#P1;
P2:=P;
GG eq sub<GG|P1,P2>;
G:=GG;

DB:=DoubleCosetRepresentatives(G,B,B);
TrP1:=Transversal(P1,B);
TrP2:=Transversal(P2,B);

for i:=1 to #DB do
if DB[i] in P1 then a:=i;end if;end for;
for i:=1 to #DB do
if DB[i] in P2 then i;end if;end for;

Index(B, B meet B^DB[i]); i=3,5,44
Index(B, B meet B^TrP2[i]) i=2:15
NeighboursofB:=[TrP1[2],TrP1[3],TrP2[4] ,TrP2[5] ,TrP2[2] ,TrP2[3] ,TrP2[10] ,
TrP2[12] ,TrP2[6] ,TrP2[7] ,TrP2[8] ,TrP2[9] ,TrP2[11] ,TrP2[13] ,TrP2[14]
,TrP2[15]];
b:=44; c:=3; d:=5;
Neighbours:=[ ];Neighbours[1]:=[a,a,b,b,c,c,c,c,d,d,d,d,d,d,d,d];

BorbitsDiscs:=[];BorbitsDiscs[1]:={a,b,c,d};
```

24.4 Code for M_{12}

```
S:=Sym(12);
K:=sub<Sym(12)| (2,3,5,9,8,10,6,11,4,7,12), (1,12)(2,11)(3,10)(4,9)(5,8)(6,7)>;
x:=Sym(12)!(7,12,5,10)(11,3,8,6,4,9)(2,1);
M12:=K^x;
G:=M12;
ans:={}; for g in G do X:={5,7,6,11}^g; if X eq {8,9,10,12} then
ans:=ans join {g}; end if; end for;

ans1:={}; for g in ans do X:={ 8,9,10,12}^g; if X eq {5,7,6,11} then
ans1:=ans1 join {g}; end if; end for;
X:=sub<G|ans1>;

H:=Stabilizer(G, {8,9,10,12}) meet Stabilizer(G, {5,7,6,11});

Z:=sub<G|XX,H>;

BB:=Z meet Stabilizer(G, {1,2,3,4});

DBB:=DoubleCosetRepresentatives(G,BB,BB);
#DBB;
B:=BB;

NB:=NormalSubgroups(B);
for i:=1 to #NB do if #NB[i]`subgroup eq 16 and IsAbelian(NB[i]`subgroup) then i;
end if; end for;
P2:=Normalizer(G,NB[16]`subgroup);
P1:=Centralizer(G, Centre(B));
G eq sub<G | P1, P2>;
#(P1 meet P2);


DB:=DoubleCosetRepresentatives(G,B,B);

TrP1:=Transversal(P1,B);
TrP2:=Transversal(P2,B);
NeighboursofB:=[TrP1[2],TrP1[3],TrP2[2],TrP2[3]];

for i:=1 to #DB do
if DB[i] in P1 then a:=i;end if;end for;
for i:=1 to #DB do
if DB[i] in P2 then b:=i;end if;end for;
```

```
Neighbours:=[ ];Neighbours[1]:=[a,a,b,b];

BorbitsDiscs:=[];BorbitsDiscs[1]:={a,b};
```

Code for generators of B

```
G<x,y>:=PermutationGroup<12|(1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)*(4,10,5,6),
(1, 12)*(2, 11)*(3, 6)*(4, 8)*(5, 9)*(7, 10)>;
#G;
ans:={}; for g in G do X:={5,7,6,11}^g; if X eq {8,9,10,12} then ans:=ans join {g};
end if; end for;

ans1:=; for g in ans do X:={ 8,9,10,12}^g; if X eq {5,7,6,11} then ans1:=ans1 join {g};
end if; end for;
```

We find which elements swap our two non fixed hexads.

```
X
X:=sub<G|ans1>;

H:=Stabilizer(G, {8,9,10,12}) meet Stabilizer(G, {5,7,6,11});
```

We find which elements fix our two non fixed hexads.

```
Z:=sub<G|XX,H>;
```

Form the group generated X and H.

```
BB:=Z meet Stabilizer(G, {1,2,3,4});
```

Find those that fix the fixed hexad and either interchanged or fix the non fixed hexads.

```
DBB:=DoubleCosetRepresentatives(G,BB,BB);
#DBB;
B:=BB;
```

B=4^2:2^2, so would like to find 2 generators order 4 and 2 generators order 2.

```
X:=Generators(B);
H:=FrattiniSubgroup(B);
ans:={}; for x in X do t:=Order(x); if t eq 2 then ans:= ans join{x}; end if; end for;
```

Find all order 2 generators.

```
#ans;
ans1:={}; for x in X do t:=Order(x); if t eq 4 then ans1:= ans1 join{x}; end if;
 end for;
```

Find all order 4 generators.

```
ans2:={}; for x in ans do t:=x; if t notin sub<B|ans1> then ans2:= ans2 join {x};
end if; end for;
```

We remove any order 2 generators that we catch by squaring order 4 generators.

```
#ans2;
```

We get only 2 order 2 (wohooo).

```
#ans1;
```

We get 9 order 4, we need to cut these down to 2.

```
H:=sub<B | ans1, ans2>;
#H;
H eq B;
```

Check that we can generate B with these elements of order 2 and 4.

```
ans2;
S:=SetToSequence(ans1);
```

We are doing just going do trial and error to find any two generators of order 2 that with our order 4s generate B.

```
H1:=sub<B| ans2, S[1], S[4]>; H1 eq B;
```

Below are the ones that work;

```
K<x,y,z,t>:=PermutationGroup<12|(1, 3)*(2, 4)*(5, 8, 7, 9)*(6, 10, 11, 12),
  (5, 12, 7, 10)*(6, 9, 11, 8), (1, 4)*(5, 6)*(7, 11)*(8, 9),
    (2, 3)*(5, 11)*(6, 7)*(8, 9)>;
K eq B;
```

This is true, so we have found 4 generators of B!

24.5 Code for M_{23}

```
S24:=Sym(24);
a1:=S24!(1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23);
b1:=S24!(16,8,15,6,11,21,18,12,23,22,20)(4,7,13,2,3,5,9,17,10,19,14);
c1:=S24!(24,1)(16,4)(8,14)(15,19)(6,10)(11,17)(21,9)(18,5)(12,3)(23,2)(22,13)(20,7);
d1:=S24!(1,4,16)(9,21,19)(5,17,11)(14,8,3)(2,13,22)(10,6,7);
M24:=sub<S24|a1,b1,c1,d1>;
M23:=Stabilizer(M24,24);
M22:=Stabilizer(M23, 15);
C:=Stabilizer(M23, 15,1,9,4,21,16,19);
x1:=S24!(4,16)(21,19)(17,11)(8,3)(23,20)(2,10)(13,7)(6,22);
x2:=S24!(1,9)(4,21)(5,14)(17,8)(2,13)(10,6)(23,7)(22,20);
x3:=S24!(18,23)(12,20)(5,2)(14,10)(17,13)(8,6)(11,22)(3,7);
x4:=S24!(18,12)(5,14)(23,20)(2,10)(17,8)(11,3)(13,6)(22,7);
x5:=S24!(18,5)(12,14)(17,11)(8,3)(23,2)(20,10)(13,22)(6,7);
x6:=S24!(5,17)(18,11)(12,3)(14,8)(2,13)(23,22)(10,6)(20,7);
B:=sub<S24|x1,x2,x3,x4,x5,x6>;
NGGB:=NormalSubgroups(B);
for i:=1 to #NGGB do if #NGGB[i]`subgroup eq 16 and
 IsElementaryAbelian(NGGB[i]`subgroup)  then i; end if; end for;
X1:=Normalizer(M23, NGGB[8]`subgroup);
X2:=Normalizer(M23, NGGB[10]`subgroup);
CompositionFactors(X1); CompositionFactors(X2);
X1 eq C;
NGGC:=NormalSubgroups(C);
for i:=1 to #NGGC do if #NGGC[i]`subgroup eq 16 and
 IsElementaryAbelian(NGGC[i]`subgroup)  then i; end if; end for;
T1:=NGGC[2]`subgroup;
CompositionFactors(T1);
A1:=Normalizer(C, Centralizer(T1,B));
CompositionFactors(A1);
N:=T1;
M:=CommutatorSubgroup(M23, N,B);
A2:=Normalizer(C,M);
MS:=MaximalSubgroups(M23);
K:=MS[3]`subgroup;
TT:=Transversal(M23,K);
ans:={}; for x in TT do Y:=K^x; if #(B  meet Y) eq #B then ans:=ans join {Y};
 end if; end for;
#ans;
X:=Random(ans);
CompositionFactors(X);
K:=M22 meet C;
ans:=Subgroups(K);
```

```
ans1:={}; for i:=1 to #ans do t:=Order(ans[i]`subgroup); if t eq 5760 then
ans1:=ans1 join {ans[i]`subgroup}; end if; end for;
T:=Random(ans1);
CompositionFactors(T);
#(B  meet T) eq #B
KK:=C meet X;
anss:=Subgroups(K);
#anss;
anss1:=[]; for i:=1 to #anss do t:=Order(anss[i]`subgroup); if t eq 1152
then anss1:=anss1 join {anss[i]`subgroup};  end if; end for;
U:=KK;
P3:=A1 meet A2 meet T;
P14:=A1 meet U;
P41:=A2 meet U;
KKK:=M22 meet X;
P7:=KKK;
B eq (P14 meet P3 meet P7);
B eq (P41 meet P3 meet P7);
Relabel Ps
P11:=P1 meet P2;
P22:=P2 meet P3;
P33:=P2 meet P3;
TrP1:=Transversal(P1,B);
TrP2:=Transversal(P2,B);
TrP3:=Transversal(P3,B);

TTrP3:=SetToSequence(TrP3);
Index(B, B meet B^TrrP3[i]);

NeighboursofB:=[TrP1[2],TrP1[3], TrP2[2],TrP2[3], TrP3[14],TrP3[7], TrP3[15],
 TrP3[12],TrP3[8], TrP3[6],  TrP3[13],TrP3[11], TrP3[10],  TrP3[9],TrP3[5],
 TrP3[4], TrP3[3],  TrP3[2] ];

for i:=1 to #DB do
if DB[i] in P1 then a:=i;end if;end for;
for i:=1 to #DB do
if DB[i] in P2 then b:=i;end if;end for;
ans:={}; for i:=1 to #DB do if DB[i] in P3 then ans:=ans join {i}; end if; end for;
ans;
Index(B, B meet B^DB[14]);
ee:=14;
dd:=805;
cc:=809;
Neighbours:=[];Neighbours[1]:=[a,a,b,b,cc,cc,dd,dd,dd,dd,ee,ee,ee,ee,ee,ee,ee,ee];
BorbitsDiscs:=[];BorbitsDiscs[1]:={a,b,cc,dd,ee};
```

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