b) $M_{n\times n}(\mathbb{C})$ over $\mathbb{R}$

$$S = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} \middle| \ \alpha, \beta \in \mathbb{C} \right\}$$

* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \bar{0} & -0 \end{pmatrix} \in S$ $\qquad \alpha = \beta = 0$

$\Rightarrow S \neq \emptyset$

* let $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix}, \begin{pmatrix} \gamma & \delta \\ \bar{\delta} & -\gamma \end{pmatrix} \in S$

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} + \begin{pmatrix} \gamma & \delta \\ \bar{\delta} & -\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+\gamma & \beta+\delta \\ \bar{\beta}+\bar{\delta} & -\alpha-\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha+\gamma & \beta+\delta \\ \overline{\beta+\delta} & -(\alpha+\gamma) \end{pmatrix} \in S$$

* $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} \in S \qquad \lambda \in \mathbb{R}$

$$\lambda \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} = \begin{pmatrix} \lambda\alpha & \lambda\beta \\ \lambda\bar{\beta} & -\lambda\alpha \end{pmatrix} = \begin{pmatrix} \lambda\alpha & \lambda\beta \\ \overline{\lambda\beta} & -\lambda\alpha \end{pmatrix} \in S$$

$\Rightarrow$ S a subspace by the subspace test

c) $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} \in S \qquad \begin{array}{l} \alpha = x+iy \\ \beta = u+iv \end{array}$

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{pmatrix} = \begin{pmatrix} x+iy & u+iv \\ u-iv & -x-iy \end{pmatrix}$$

$$= \cancel{(x+iy)}\, x \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$+ u \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + v \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ span S

let $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$

$$\lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$a_{11} \Rightarrow \lambda_1 + i\lambda_2 = 0 + 0i$

$\Rightarrow \lambda_1 = \lambda_2 = 0$

$a_{21} \Rightarrow \lambda_3 - \lambda_4 i = 0 + 0i$

$\Rightarrow \lambda_3 = \lambda_4 = 0$

$\Rightarrow$ these 4 matrices are LI

$\Rightarrow$ also a basis

d) $\dim(S) = |\text{basis}| = 4$

e) let $u = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \in S$ let $\lambda = 1 + i \in \mathbb{C}$

$$\lambda u = (1+i) \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & (1+i)i \\ (1+i)(-i) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1+i \\ 1-i & 0 \end{pmatrix}$$

$$\overline{-1+i} = -1 - i$$

$\Rightarrow \lambda u \notin S$

$\Rightarrow S$ not a SS over $\mathbb{C}$ as not closed under scalar multiplication